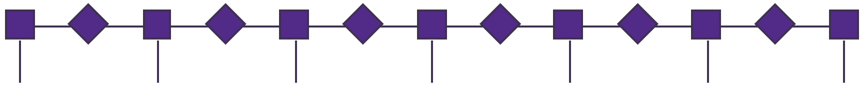
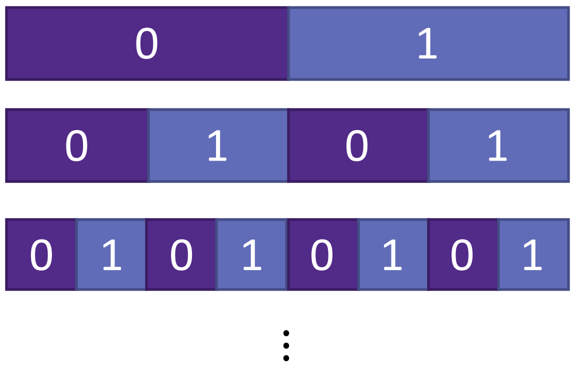
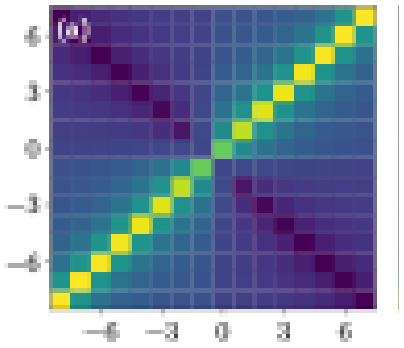
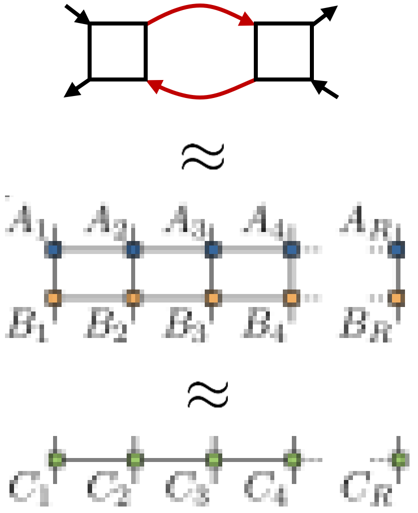
Learning tensor networks with tensor cross interpolation: New algorithms and libraries

Marc K. Ritter | LMU Munich | CCQ, Flatiron Institute | mritter@flatironinstitute.org







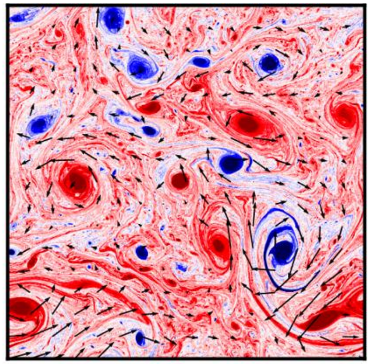


Ritter, Núñez Fernández, Wallerberger, von Delft, Shinaoka, Waintal, PRL 2024
Nuñez Fernández, Ritter, Jeannin, Li, Kloss, Louvet, Terasaki, Parcollet, von Delft, Shinaoka, Waintal: SciPost Phys. 2025

Function representation with Tensor Trains

mritter@flatironinstitute.org

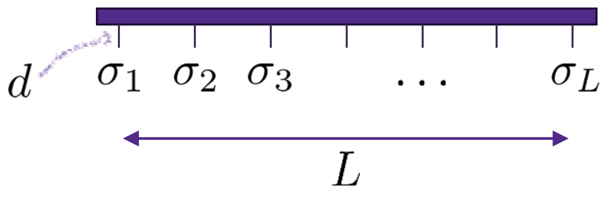
Hölscher ... Wilhelm:
arXiv:2406.17823



function
$f(x_1, x_2, \ldots, x_L)$

discretization

tensor
$F(\sigma_1, \sigma_2, \ldots, \sigma_L)$

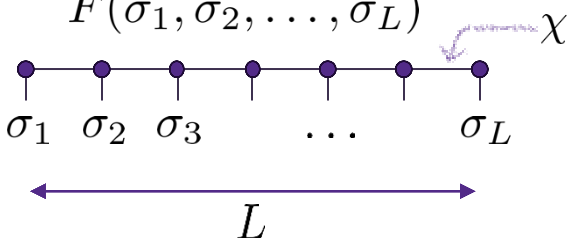


$O(d^L)$

- Full information
- Exact
- Exponential cost in L

factorization

tensor train
$\tilde{F}(\sigma_1, \sigma_2, \ldots, \sigma_L)$



$O(d\chi^2 L)$

- (Approximately) factorized
- Tradeoff χ vs. error
- Linear cost in L

Quantics Representation

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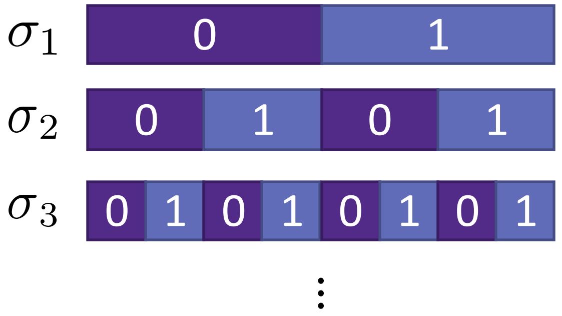
I. V. Oseledets, Dokl Math 2009; B. N. Khoromskij, Constr Approx 2011
Shinaoka, Wallerberger, Murakami, Nogaki, Sakurai, Werner, Kauch, PRX 2023
Ritter, Núñez Fernández, Wallerberger, von Delft, Shinaoka, Waintal, PRL 2024



$$x = \frac{\sigma_1}{2^1} + \frac{\sigma_2}{2^2} + \ldots + \frac{\sigma_R}{2^R}; \qquad \sigma_1, \ldots, \sigma_R \in \{0, 1\}$$

$$f(x) \Longrightarrow f(\sigma_1, \ldots, \sigma_R) = f\left(\tfrac{\sigma_1}{2^1} + \tfrac{\sigma_2}{2^2} + \ldots + \tfrac{\sigma_R}{2^R}\right)$$

$$=: f_\sigma =$$

Large scale ⟷ Small scale

Example: exponential

$$\exp(ax) = \exp\left[a\left(\tfrac{\sigma_1}{2^1} + \tfrac{\sigma_2}{2^2} + \tfrac{\sigma_3}{2^3} + \ldots + \tfrac{\sigma_R}{2^R}\right)\right]$$

$$= \exp\left(\tfrac{a}{2^1}\sigma_1\right) \exp\left(\tfrac{a}{2^2}\sigma_2\right) \exp\left(\tfrac{a}{2^3}\sigma_3\right) \times \ldots \times \exp\left(\tfrac{a}{2^R}\sigma_R\right)$$

$\chi = 1$ for exponential $\exp(ax)$
$\chi = n$ for superposition $\sum_{i=1}^{n} b_i \exp(a_i x)$

Multivariate quantics representation

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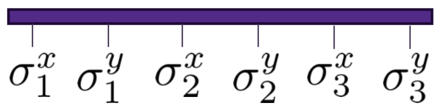
I. V. Oseledets, Dokl Math 2009; B. N. Khoromskij, Constr Approx 2011
Shinaoka, Wallerberger, Murakami, Nogaki, Sakurai, Werner, Kauch, PRX 2023
Ritter, Núñez Fernández, Wallerberger, von Delft, Shinaoka, Waintal, PRL 2024

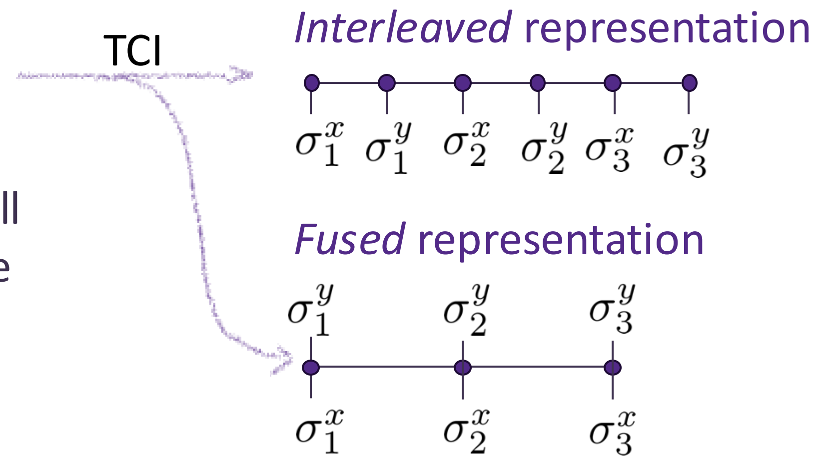
$x, y \in [0, 1]$

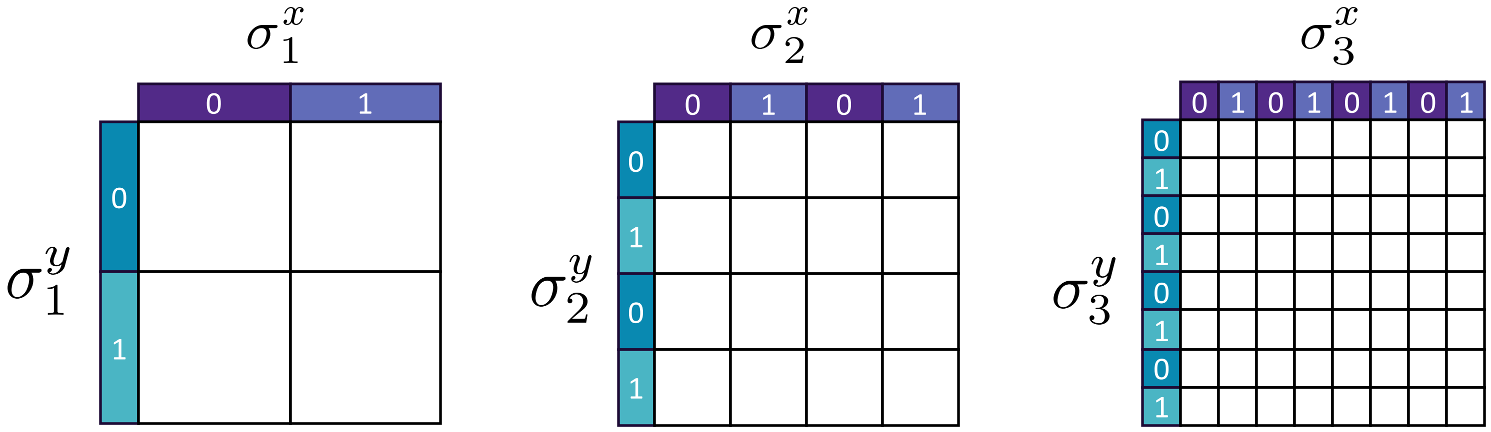
$\hookrightarrow \; x = \frac{\sigma_1^x}{2^1} + \frac{\sigma_2^x}{2^2} + \ldots + \frac{\sigma_R^x}{2^R}; \quad y = \frac{\sigma_1^y}{2^1} + \frac{\sigma_2^y}{2^2} + \ldots + \frac{\sigma_R^y}{2^R}; \quad \sigma_\ell^{x,y} \in 0, 1$

$f(x, y) \Rightarrow f(\sigma_1^x, \ldots, \sigma_R^x, \sigma_1^y, \ldots, \sigma_R^y) = f_{\boldsymbol{\sigma}} =$



Large scale $\longleftrightarrow$ Small scale





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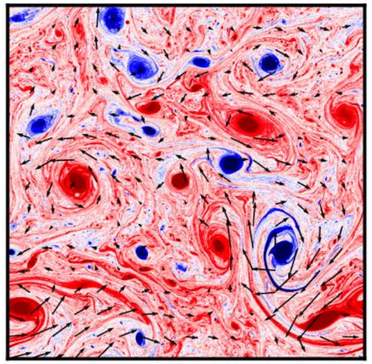
Function representation with Tensor Trains

Hölscher ... Wilhelm: arXiv:2406.17823



function
$f(x_1, x_2, \ldots, x_L)$

discretization

tensor
$F(\sigma_1, \sigma_2, \ldots, \sigma_L)$

d σ_1 σ_2 σ_3 $\ldots$ σ_L

L

$O(d^L)$

- Full information
- Exact
- Exponential cost in L

factorization

tensor train
$\tilde{F}(\sigma_1, \sigma_2, \ldots, \sigma_L)$ χ

σ_1 σ_2 σ_3 $\ldots$ σ_L

L

$O(d\chi^2 L)$

- (Approximately) factorized
- Tradeoff χ vs. error
- Linear cost in L

2-site tensor cross interpolation

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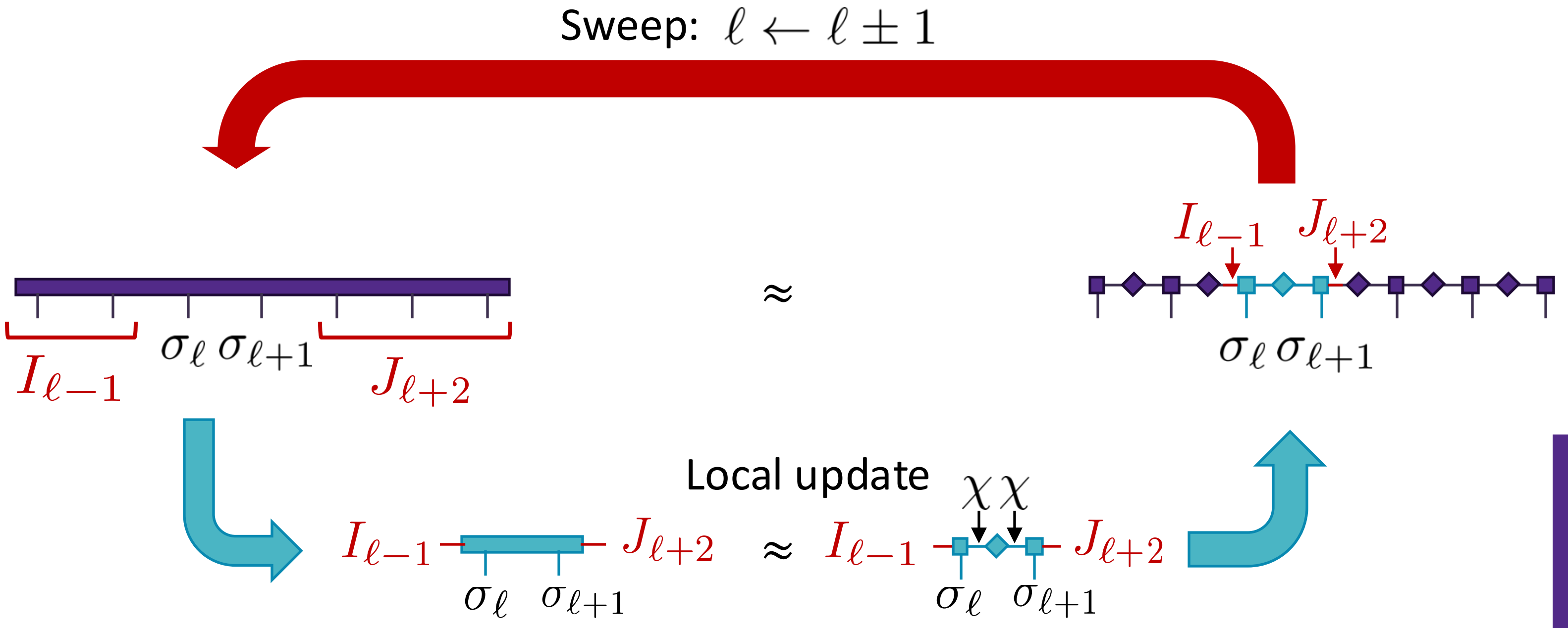
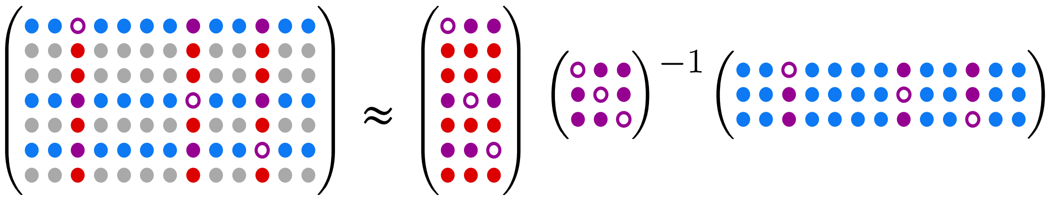
Oseledets, SIAM J Sci Comput 2011; Oseledets, Tyrtyshnikov, Lin Alg Appl 2011
Nuñez Fernández, Jeannin, Dumitrescu, Kloss, Kaye, Parcollet, Waintal, PRX 2022
Nuñez Fernández, Ritter, Jeannin, Li, Kloss, Louvet, Terasaki, Parcollet, von Delft, Shinaoka, Waintal: SciPost Phys. 18 (2025)



$$\Pi_{i\sigma_\ell;\sigma_{\ell+1}j} \approx \sum_{ab} T_{i\sigma_\ell;a} P^{-1}_{ab} T_{b;\sigma_{\ell+1}j}$$



Runtime: $O(\chi^3 d^2 L)$
Memory: $O(\chi^2 d^2 L)$

CI-canonical form

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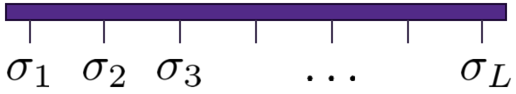
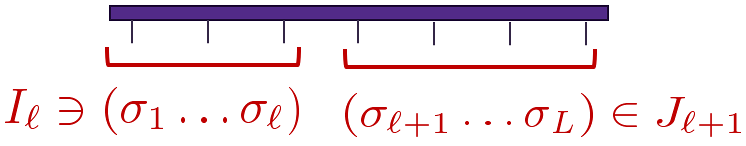
Nuñez Fernández, Ritter, Jeannin, Li, Kloss, Louvet, Terasaki, Parcollet, von Delft, Shinaoka, Waintal: SciPost Phys. 18 (2025)

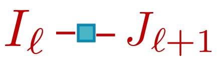
Full tensor: $F(\sigma_1, \sigma_2, \ldots, \sigma_L)$



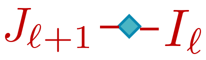
Slices:

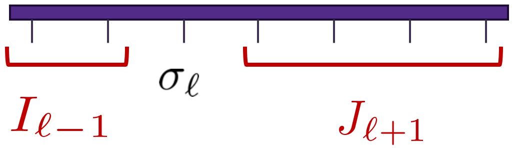


$P_\ell = F(I_\ell, J_{\ell+1})$

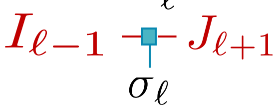


P_ℓ^{-1}

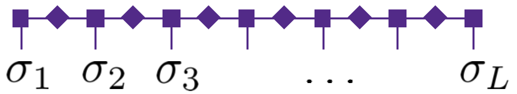




$T_\ell^{\sigma_\ell} = F(I_{\ell-1}, \sigma_\ell, J_{\ell+1})$



$T_1^{\sigma_1} P_1^{-1} T_2^{\sigma_2} P_2^{-1} T_3^{\sigma_3} \ldots T_L^{\sigma_L}$



Example

$$I_3 = \{(1,1,2), (1,2,2)\}$$
$$J_4 = \{(1,1), (2,1)\}$$

$$\begin{array}{cc} & \begin{array}{cc} \multicolumn{2}{c}{J_4} \\ (1,1)\downarrow & (2,1)\downarrow \end{array} \\ I_3 \begin{array}{c} (1,1,2)\rightarrow \\ (1,2,2)\rightarrow \end{array} & \begin{bmatrix} F(1,1,2,1,1) & F(1,1,2,2,1) \\ F(1,2,2,1,1) & F(1,2,2,2,1) \end{bmatrix} \end{array} = F(I_3, J_4) = P_3$$

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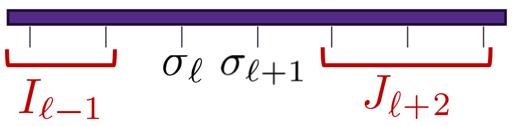
Evaluating the error

Nuñez Fernández, Ritter, Jeannin, Li, Kloss, Louvet, Terasaki, Parcollet, von Delft, Shinaoka, Waintal: SciPost Phys. 18 (2025)

Global error =



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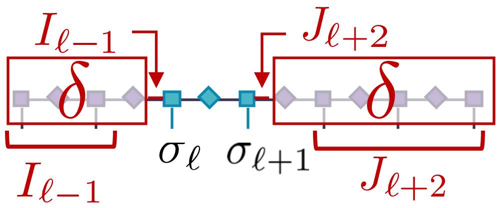
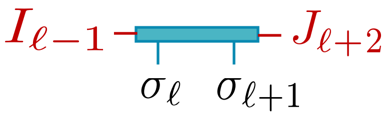


def

nesting

Local error =



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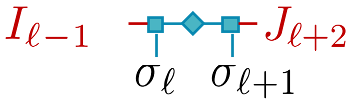
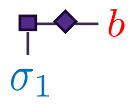
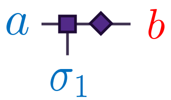


Tails



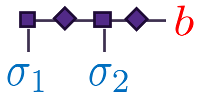
$$\left(T_1^{\sigma_1} P_1^{-1}\right)_b = \left(F(\sigma_1, J_2)\,[F(I_1, J_2)]^{-1}\right)_b \qquad = \delta_{\sigma_1; I_1(b)}$$

nesting condition



$$\left(T_\ell^{\sigma_\ell} P_\ell^{-1}\right)_{a;b} = \left(F(I_{\ell-1}, \sigma_\ell, J_{\ell+1})\,[F(I_\ell, J_{\ell+1})]^{-1}\right)_{a;b} = \delta_{I_{\ell-1}(a), \sigma_\ell; I_\ell(b)}$$

if $I_{\ell-1}(a) \oplus \sigma_\ell \in I_\ell$



$$\left(T_1^{\sigma_1} P_1^{-1} T_2^{\sigma_2} P_2^{-1} \ldots T_\ell^{\sigma_\ell} P_\ell^{-1}\right)_b \qquad = \delta_{\sigma_1 \sigma_2 \ldots \sigma_\ell; I_\ell(b)}$$

if nesting condition holds for all substrings $\sigma_1 \ldots \sigma_\ell$,

On $I_{\ell-1} \times \{\sigma_\ell \sigma_{\ell+1}\} \times J_{\ell+2}$,
global error = local error

2-site TCI algorithms

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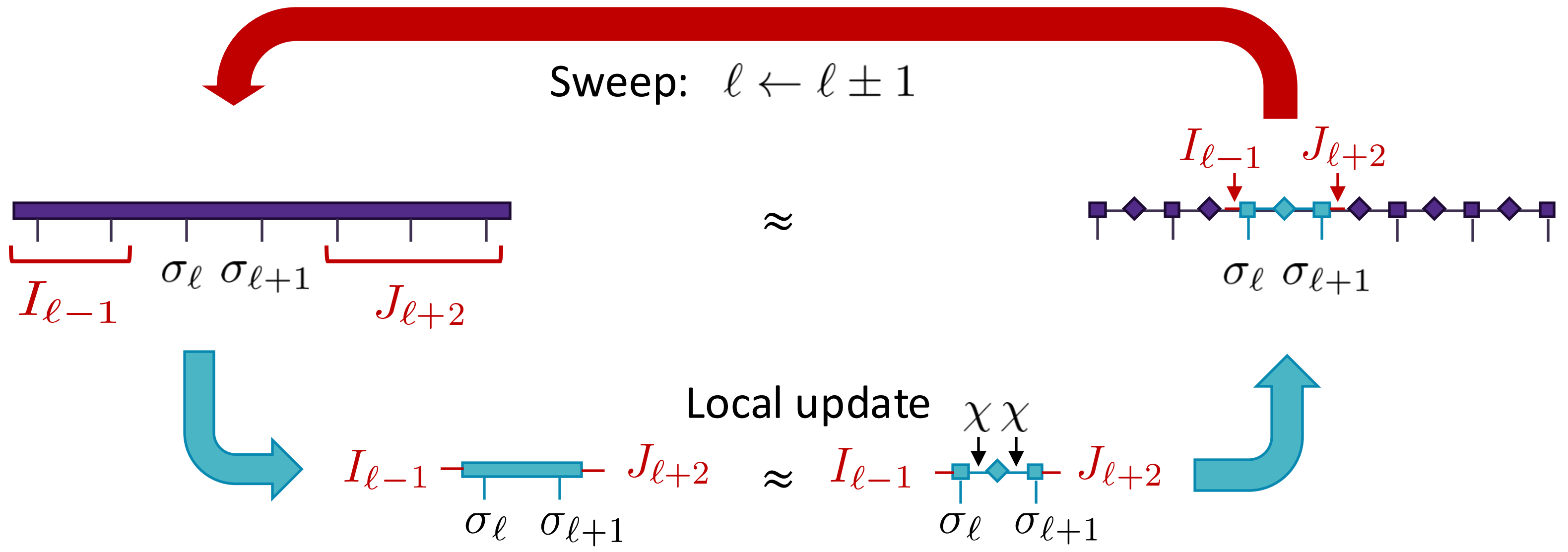
Nuñez Fernández, Ritter, Jeannin, Li, Kloss, Louvet, Terasaki, Parcollet, von Delft, Shinaoka, Waintal: SciPost Phys. 18 (2025)



Local update options	**Old / TCI1**	**New / TCI2** (Recommended default)
Factorization	CI	prrLU
Pivot updates	Accumulate χ sweeps adding 1 pivot each	Reset n_{Sweep} sweeps adding/removing $\leq \chi$ pivots each
Pivot search	Full $O(d^2\chi^3)$	Block rook $O(d\chi^3)$

Partial rank-revealing LU

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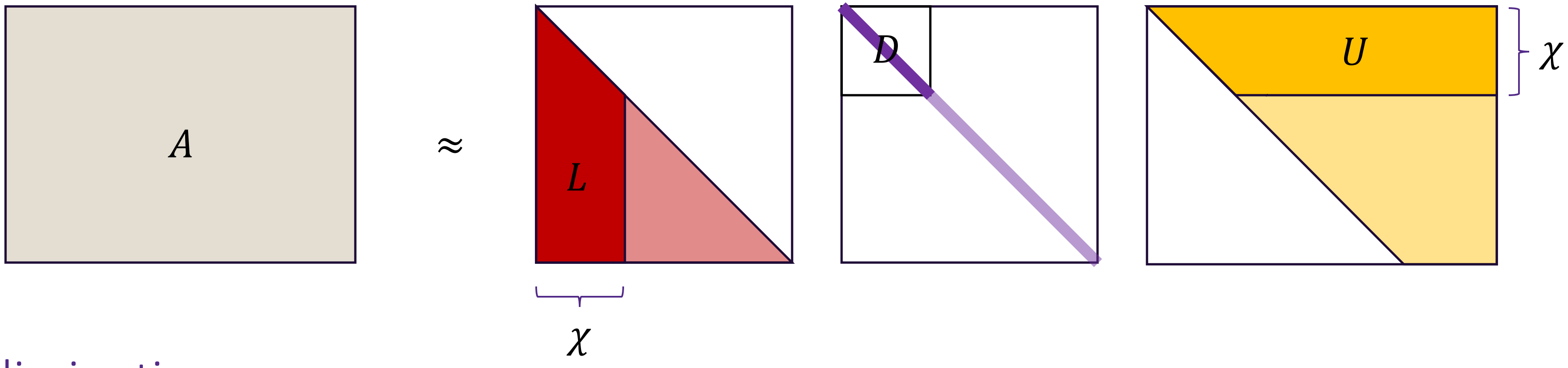
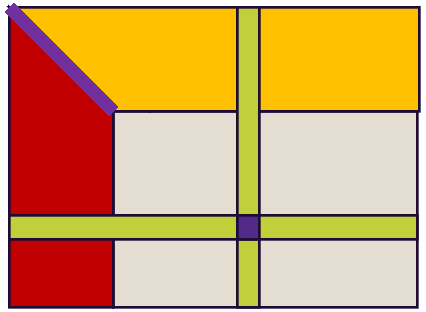
Nuñez Fernández, Ritter, Jeannin, Li, Kloss, Louvet, Terasaki, Parcollet, von Delft, Shinaoka, Waintal: SciPost Phys. 18 (2025)

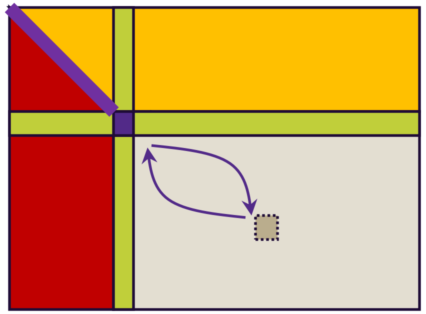


Gaussian elimination

1. Find new pivot



2. Swap rows & columns



3. Elimination

4. Iterate until next pivot < tolerance, or $\chi = \chi_{\max}$

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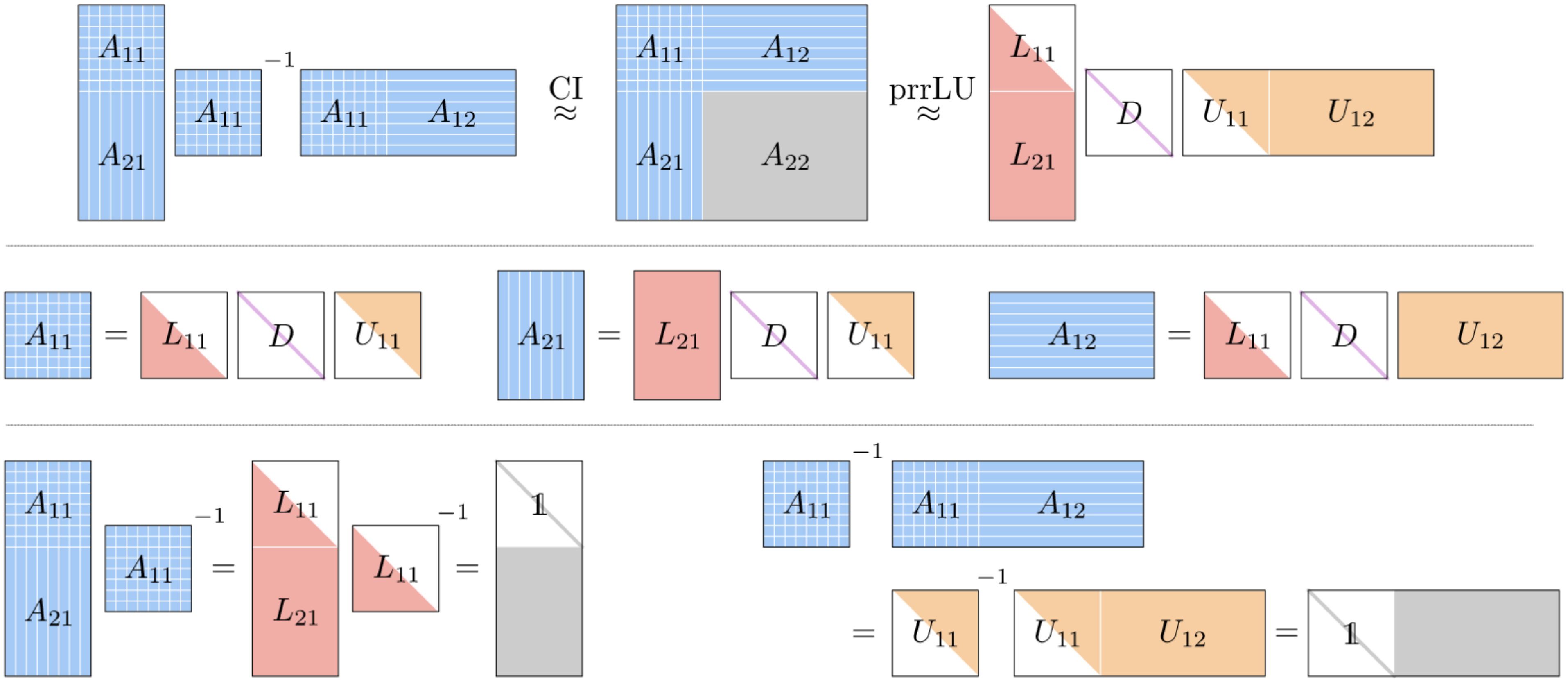
CI factorization from LU

Nuñez Fernández, Ritter, Jeannin, Li, Kloss, Louvet, Terasaki, Parcollet, von Delft, Shinaoka, Waintal: SciPost Phys. 18 (2025)



How to find good pivots

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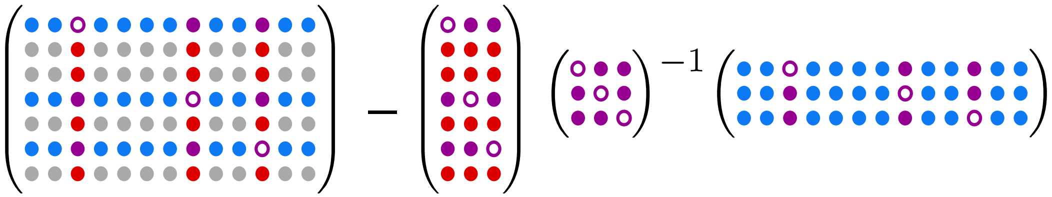
Nuñez Fernández, Ritter, Jeannin, Li, Kloss, Louvet, Terasaki, Parcollet, von Delft, Shinaoka, Waintal: SciPost Phys. 18 (2025)

What is a good pivot?

$A - \tilde{A} =$ 

$= A - CP^{-1}R$

$= [A/P]$ (Schur complement)

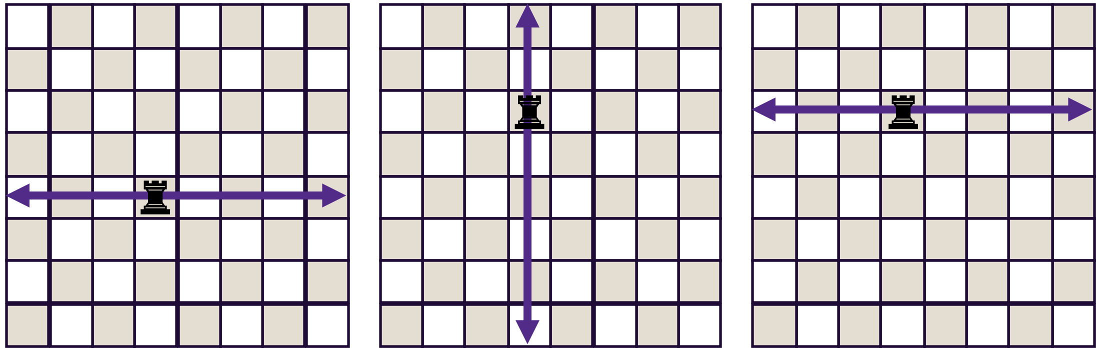
- *Intuitively:*
  Eliminate maximum error $\max(|A - \tilde{A}|)$
- *Maxvol principle:*
  $\tilde{A}$ is quasi-optimal if $|\det P|$ is maximal
  Goreinov, Tyrtyshnikov: Dokl. Math. (2011)

Algorithms

Full search $O(mn)$

1. Generate $[A/P]$
2. Look for maximum

Rook search $O(\max(m,n))$

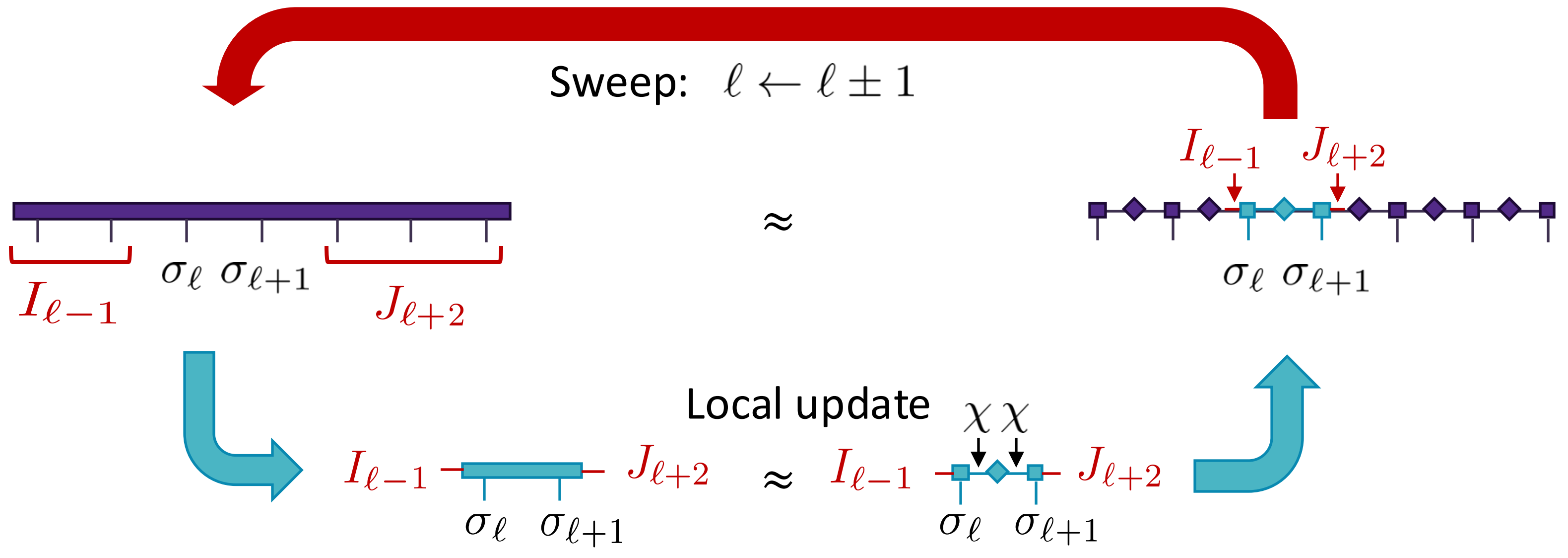


Block rook search $O(\chi \max(m,n)/\chi)$

1. Get column matrix C
2. Update pivots with full search
3. Get row matrix R
4. Update pivots with full search
5. Iterate 1 – 4 to convergence

2-site TCI algorithms

Nuñez Fernández, Ritter, Jeannin, Li, Kloss, Louvet, Terasaki, Parcollet, von Delft, Shinaoka, Waintal: SciPost Phys. 18 (2025)



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CI-canonical form

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Nuñez Fernández, Ritter, Jeannin, Li, Kloss, Louvet, Terasaki, Parcollet, von Delft, Shinaoka, Waintal: SciPost Phys. 18 (2025)

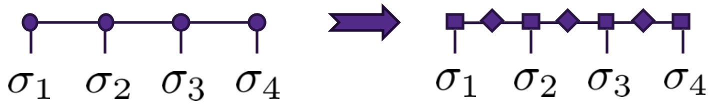
Uses

- Interpolation property
- Pivot information
- Re-using pivots

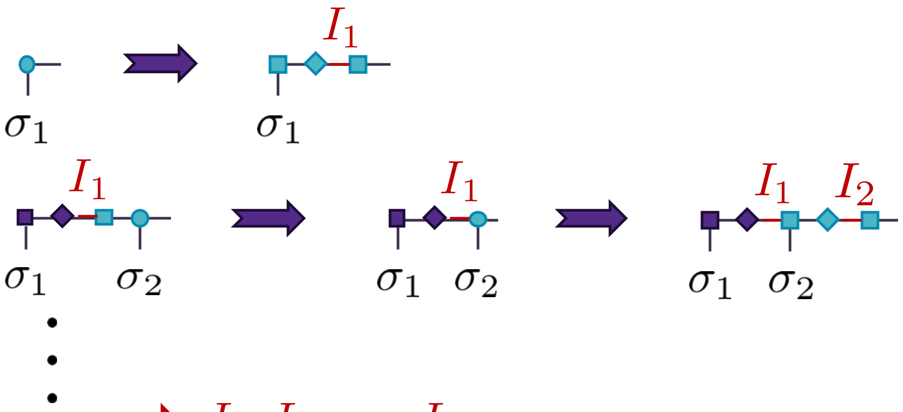
Obtaining CI-canonical TT

- Tensor Cross Interpolation (2-site, 1-site, 0-site)
- CI-canonicalization
- Global pivots

CI-canonicalization



1. Forward sweep: Generate I sets



$\rightarrow I_1, I_2, \ldots, I_L$

2. Backward sweep: Generate J sets

$\rightarrow J_1, J_2, \ldots, J_L$

3. Forward sweep: Restore full nesting

Stabilizing convergence

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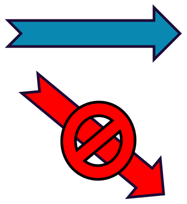
Reachable by local updates

$f(\boldsymbol{\sigma}), \boldsymbol{\sigma} \in \{1, \dots, 10\}^5$

[1, 1, 1, 1, 1]



[2, 3, 1, 1, 1], [1, 2, 2, 1, 1],
[1, 1, 5, 9, 1], [1, 1, 1, 3, 5]

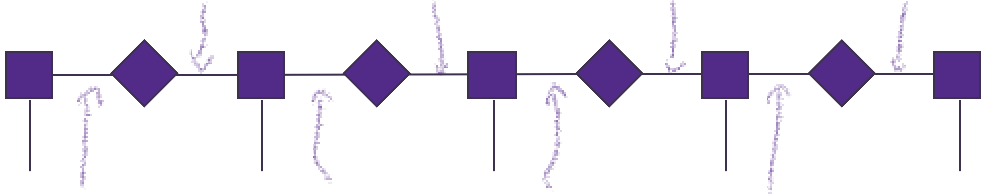
[8, 3, 5, 1, 1], [1, 2, 1, 1, 2]

Adding global pivots

Example: add [1, 2, 3, 4, 5]

$$P_\ell = F(I_\ell, J_{\ell+1})$$
$$T_\ell^{\sigma_\ell} = F(I_{\ell-1}, \sigma_\ell, J_{\ell+1})$$

Add to I_ℓ [1] [1, 2] [1, 2, 3] [1, 2, 3, 4]

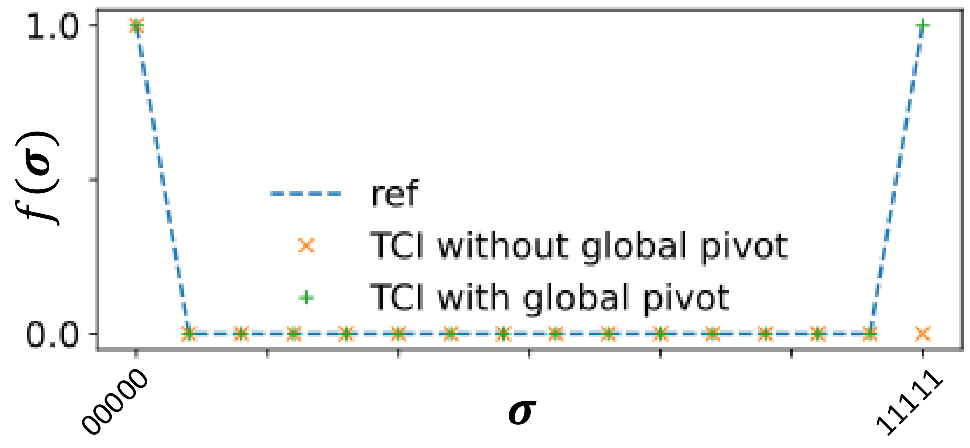


Add to J_ℓ [2, 3, 4, 5] [3, 4, 5] [4, 5] [5]

Example

$f: \{0,1\}^5 \to \{0,1\}$

$\boldsymbol{\sigma} \mapsto \delta_{\boldsymbol{\sigma},00000} + \delta_{\boldsymbol{\sigma},11111}$



Automatic search

1. Generate random $\boldsymbol{\sigma}$
2. Global rook search for maximum $f(\boldsymbol{\sigma})$
3. Add $\boldsymbol{\sigma}$ as global pivot

Map of libraries

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Julia library maintainers:
Hiroshi Shinaoka and Marc Ritter

xfac
C++/Python
Yuriel Núñez Fernández, Xavier Waintal

Quantics

PolynomialQTT.jl
[Lindsey 2024]

QuanticsTCI.jl
Convenient interface

QuanticsGrids.jl
Conversions to/from quantics indices

QuanticsTransforms.jl
Quantics Fourier Transform, affine transforms

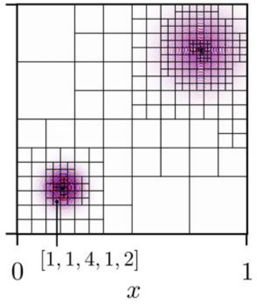
Tensor Trains

TensorCrossInterpolation.jl
TCI, rank-revealing LU

TensorTrains.jl
TT & TT contractions

ProjMPS.jl
Adaptive patching

Gianluca Grosso



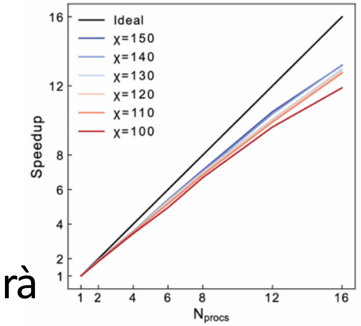
Parallelization

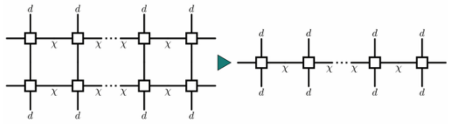
ParallelTCI.jl
Parallel TCI

MPI

ParallelTensorTrains.jl
Parallel contractions

Simone Foderà

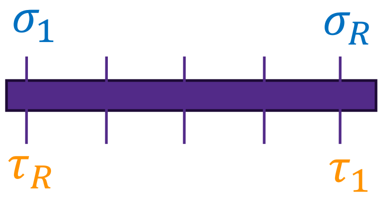




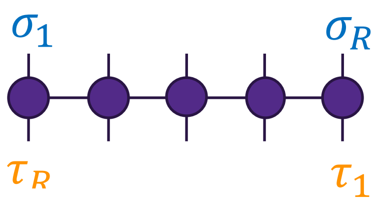
Quantics Fourier transform

Dolgov, Khoromskij, Savostyanov: J Fourier Analysis Appl (2012)
Chen, Stoudenmire, White: PRX Quantum (2023)
Chen, Lindsey: arxiv:2404.03192

$$F = \exp\left(-\frac{2\pi i}{N}\, k\, x\right) = \exp\left(-i\pi \sum_{\ell,\ell'=1}^{n} 2^{R-\ell-\ell'} \tau_\ell \sigma_{\ell'}\right) =$$



$\approx$



Quantum Fourier Transform Has Small Entanglement

Jielun Chen (陈捷伦)[1,2,*] E.M. Stoudenmire[3] and Steven R. White[1]
[1] *Department of Physics and Astronomy, University of California, Irvine, California 92697-4575, USA*
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(Received 2 January 2023; accepted 25 September 2023; published 27 October 2023)

Direct interpolative construction of the discrete Fourier transform as a matrix product operator

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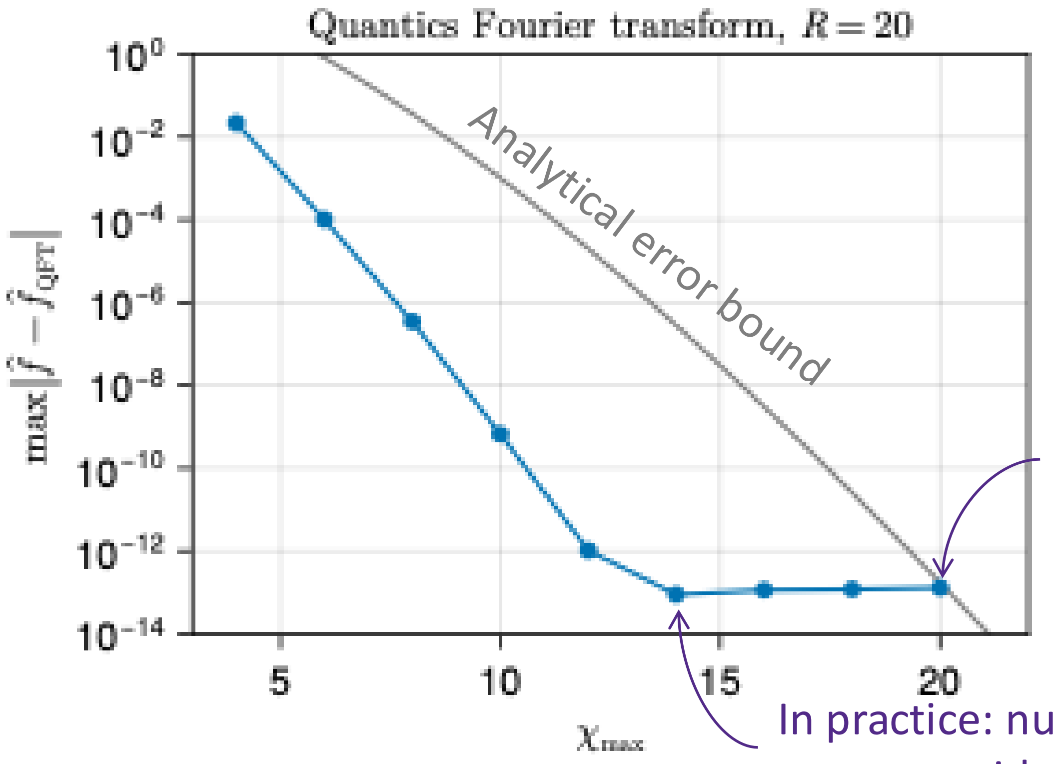
April 5, 2024



Analytical bound: numerical accuracy with $\chi = 20$

In practice: numerical accuracy with $\chi = 14$

Example: Diagrammatics for SYK model

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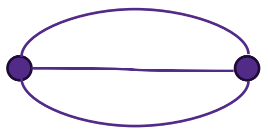
$$H_4 = \frac{1}{(2N)^{3/2}} \sum_{ijk\ell=1}^{N} U_{ij;k\ell}\, c_i^\dagger c_j^\dagger c_k c_\ell - \mu \sum_i c_i^\dagger c_i$$

random

Disorder-averaged self-energy diagram:

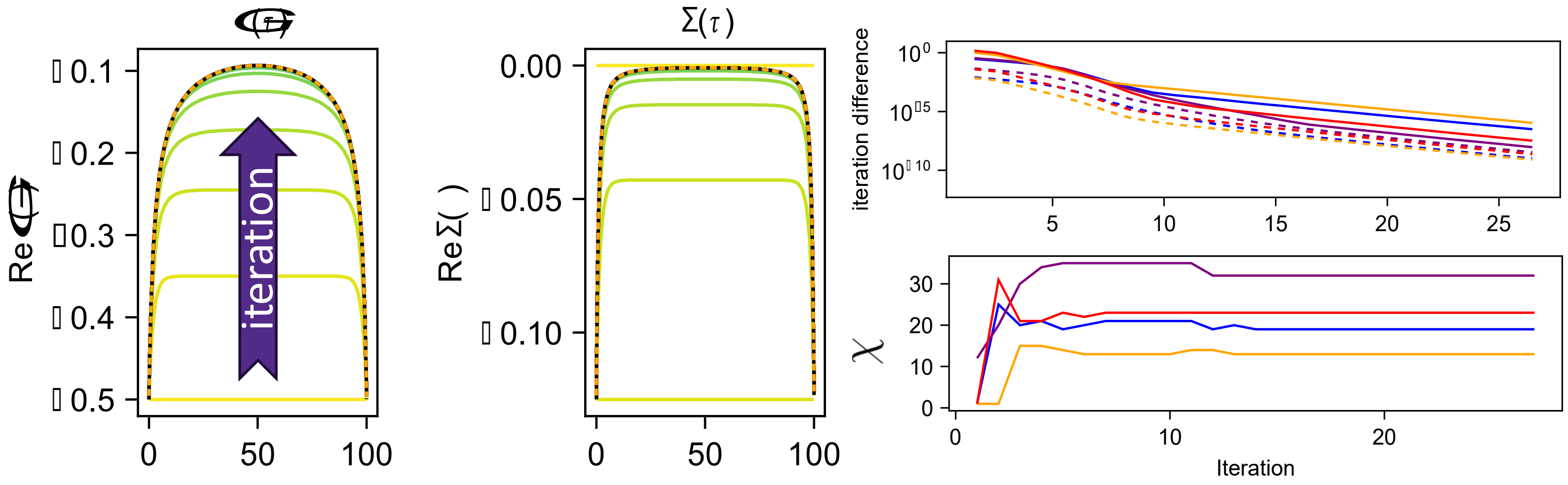


$$\Sigma(\tau) = J^2[G(\tau)]^2 G(\beta-\tau)$$

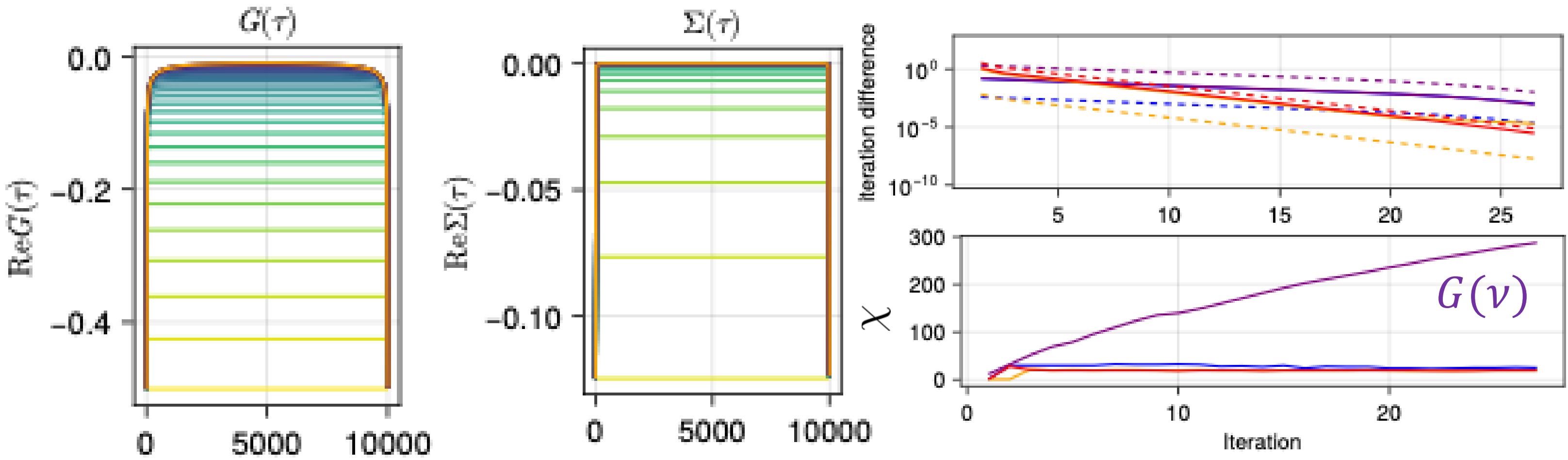
FT^{-1} Self-consistency iteration FT

$$G(i\nu) = \frac{1}{i\nu + \mu - \Sigma(i\nu)}$$

$\beta = 10^2$



$\beta = 10^4$



Project: QTT parquet equation solver

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Rohshap, Ritter, Shinaoka, von Delft, Wallerberger, Kauch, PRR 2025

Stefan Rohshap

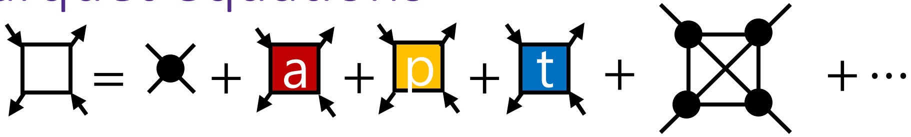
Hiroshi Shinaoka

Jan von Delft

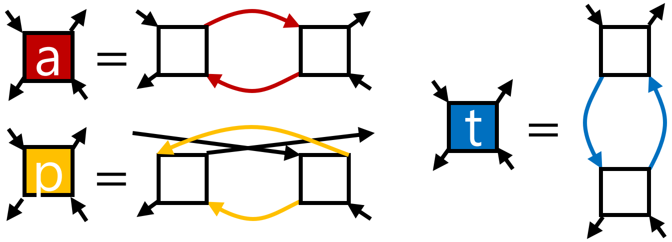
Markus Wallerberger

Anna Kauch

Parquet equations



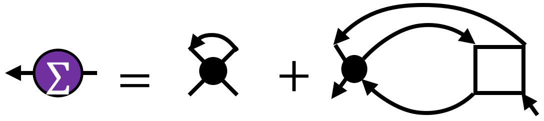
Bethe–Salpeter equations

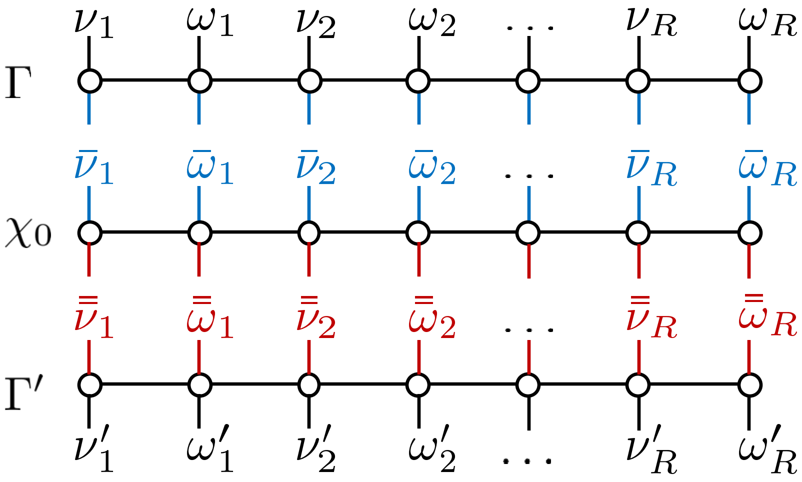


$$\sim \sum_{\bar\nu\bar{\bar\nu}} \Gamma(\nu,\bar\nu,\omega)\chi_0(\bar\nu,\bar{\bar\nu},\omega)\Gamma'(\bar{\bar\nu},\nu',\omega)$$

$$= \sum_{\bar\nu,\bar\omega}\sum_{\bar{\bar\nu},\bar{\bar\omega}} \Gamma(\nu,\omega;\bar\nu,\bar\omega)\chi_0(\bar\nu,\bar\omega;\bar{\bar\nu},\bar{\bar\omega})\Gamma'(\bar{\bar\nu},\bar{\bar\omega};\nu',\omega')\Big|_{\omega'=\omega}$$

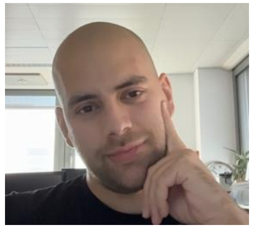
Schwinger–Dyson equation





Runtime: $O(\chi^4 L)$

Memory: $O(\chi^2 L)$



Samuel Badr – Poster:
BSE with momentum dependence

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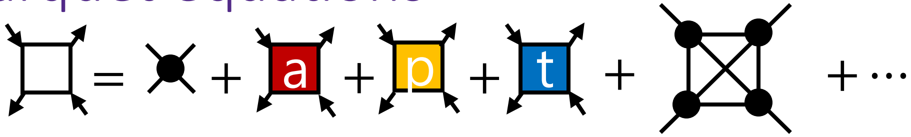
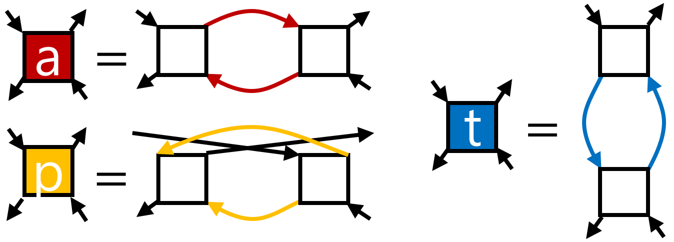
Rohshap, Ritter, Shinaoka, von Delft, Wallerberger, Kauch, PRR 2025

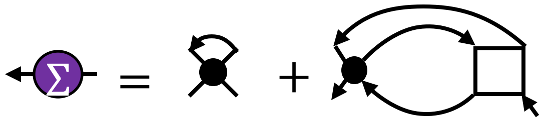
Parquet equations



Bethe–Salpeter equations



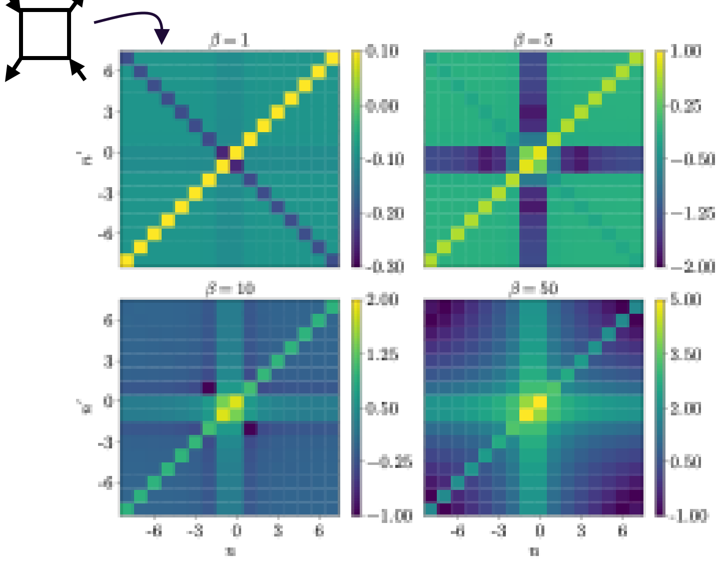
Schwinger–Dyson equation

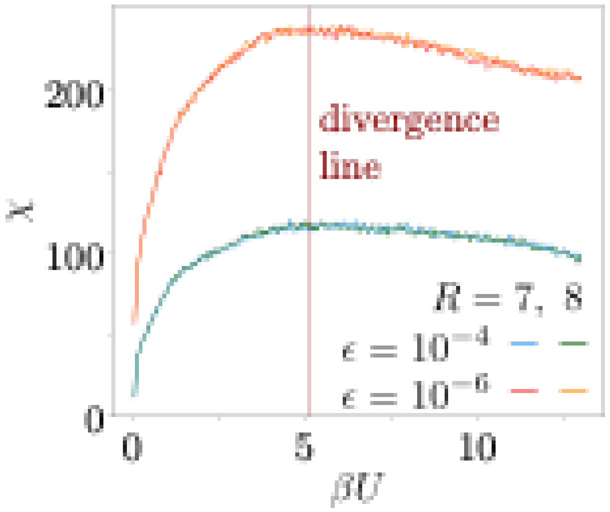


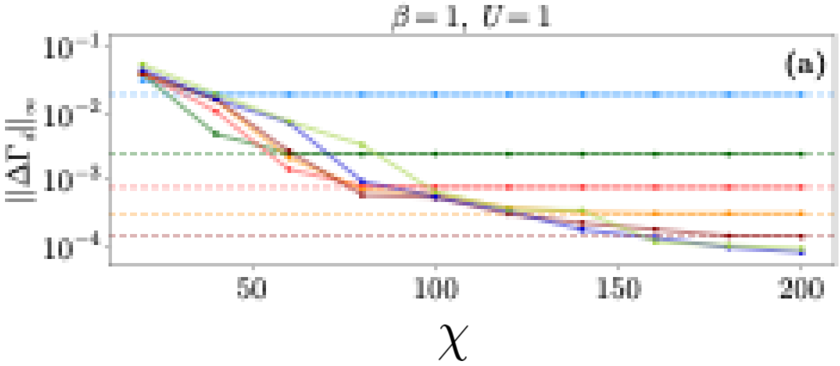
Hubbard atom

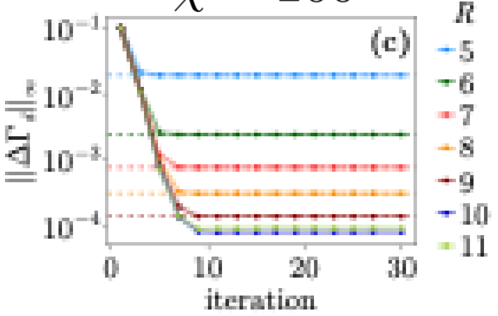
$$\hat{\mathcal{H}} = U\hat{n}_\uparrow \hat{n}_\downarrow - \mu(\hat{n}_\uparrow + \hat{n}_\downarrow)$$

Exact data:
Thunström, Gunnarsson, Ciuchi, Rohringer, PRB 2018









Project: QTT parquet equation solver

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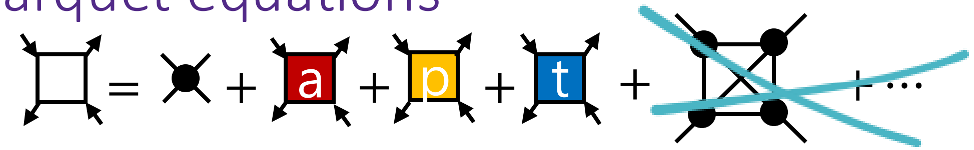
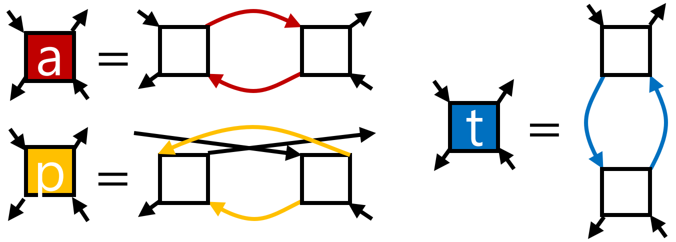
Rohshap, Ritter, Shinaoka, von Delft, Wallerberger, Kauch, PRR 2025

Parquet equations

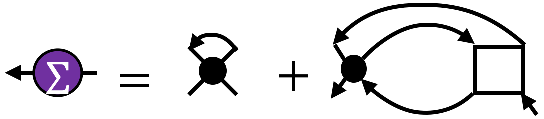


Parquet approx.

Bethe–Salpeter equations



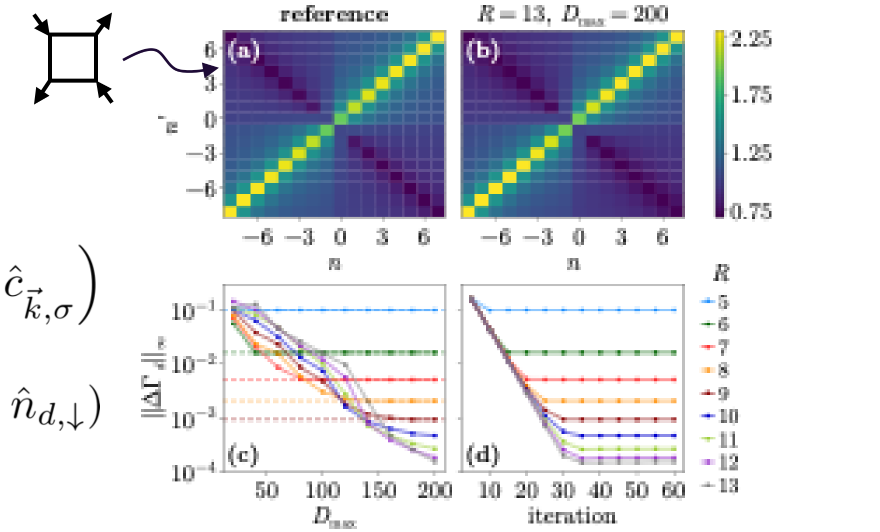
Schwinger–Dyson equation



Single-impurity Anderson model (SIAM)

Reference data:
Krien, Kauch, EPJ B 2022

$$
\begin{aligned}
\hat{\mathcal{H}} &= \sum_{\vec{k}\sigma} \varepsilon_{\vec{k}} \hat{c}^\dagger_{\vec{k},\sigma} \hat{c}_{\vec{k},\sigma} \\
&+ \sum_{\vec{k}\sigma} \left(V_{\vec{k}} \hat{c}^\dagger_{\vec{k},\sigma} \hat{d}_\sigma + V^*_{\vec{k}} \hat{d}^\dagger_\sigma \hat{c}_{\vec{k},\sigma} \right) \\
&+ U \hat{n}_{d,\uparrow} \hat{n}_{d,\downarrow} + \varepsilon_d (\hat{n}_{d,\uparrow} + \hat{n}_{d,\downarrow})
\end{aligned}
$$

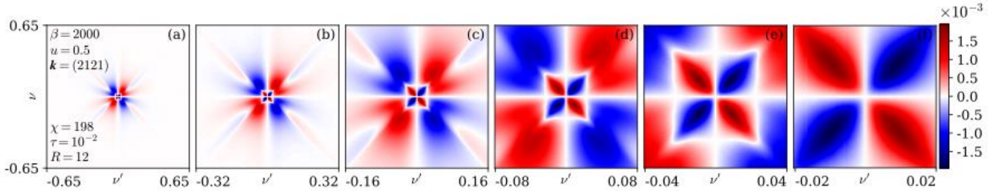


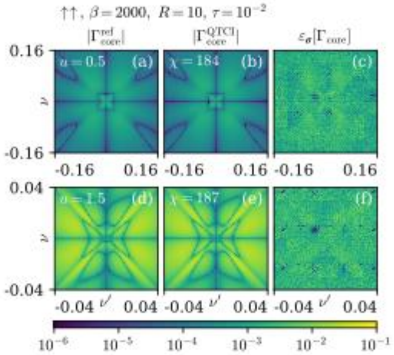
χ

Frankenbach, Ritter, Pelz, Ritz, von Delft, Ge: arxiv:2506.13359

QTT-DΓA

Local vertex from DMFT as input
→ Poster Markus Frankenbach





Summary

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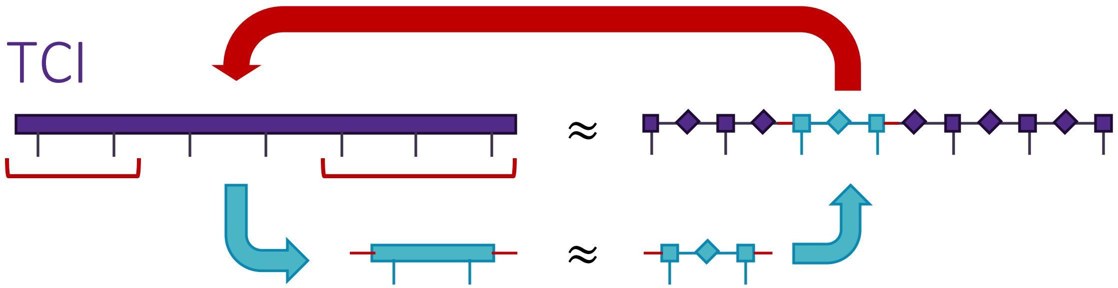
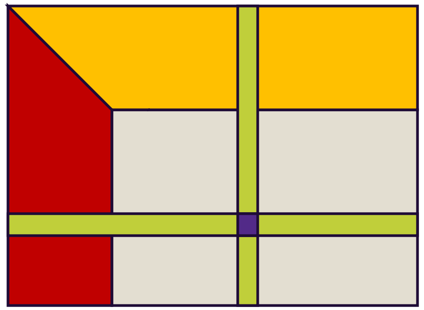
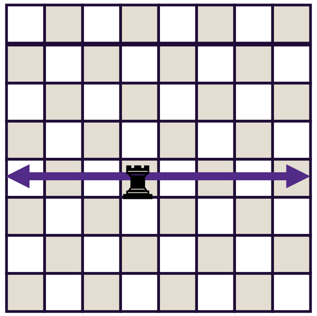
TCI



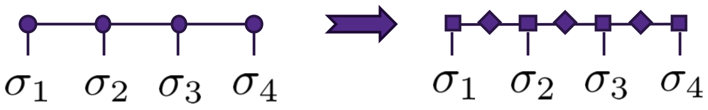
prrLU



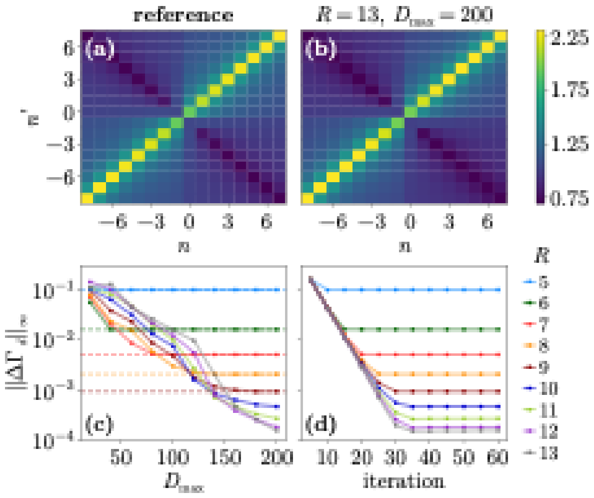
Rook pivoting



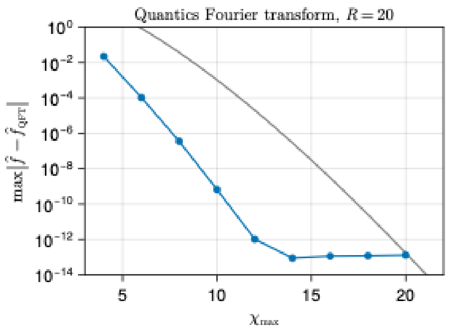
CI-canonicalization



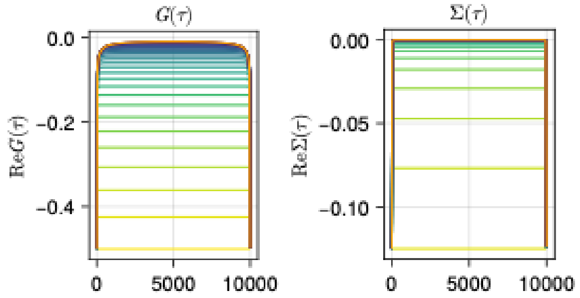
QTT-Parquet



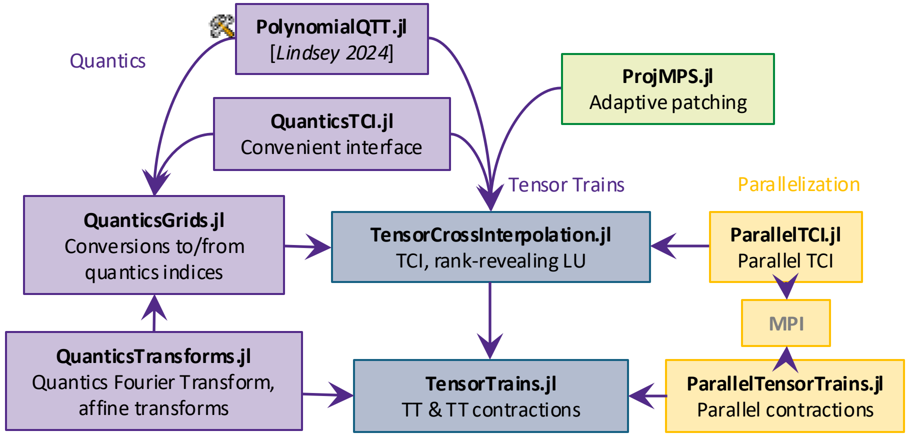
Fourier transform



SYK model



Libraries

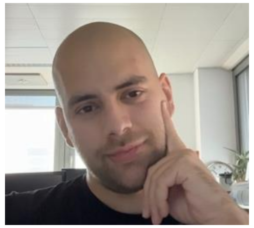


Thank you for your attention!

Yuriel Núñez Fernández

Markus Wallerberger

Hiroshi Shinaoka



Samuel Badr

Stefan Rohshap

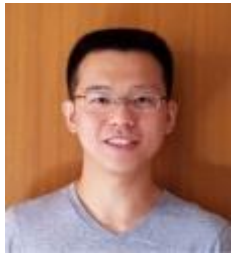
Anna Kauch

Jan von Delft

Gianluca Grosso

Xavier Waintal

Olivier Parcollet



Jheng-Wei Li

Simone Foderà

Markus Frankenbach

Satoshi Terasaki Thomas Kloss

Matthieu Jeannin Thibaud Louvet

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Interpolation property

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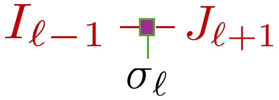
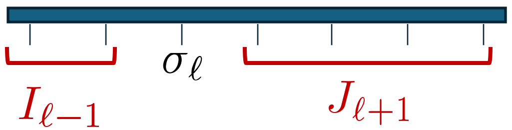
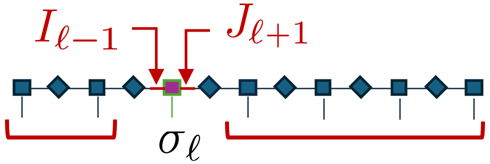
$$T_\ell^{\sigma_\ell} \quad = \quad F(I_{\ell-1}, \sigma_\ell, J_{\ell+1}) \quad \stackrel{?}{=} \quad T_1^{\sigma_1} P_1^{-1} T_2^{\sigma_2} P_2^{-1} \dots T_\ell^{\sigma_\ell} P_\ell^{-1} T_{\ell+1}^{\sigma_{\ell+1}} \dots T_L^{\sigma_L}$$



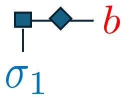
=



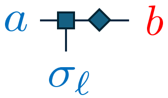
$\stackrel{?}{=}$



Tails

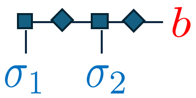


$$\left(T_1^{\sigma_1} P_1^{-1}\right)_b = \left(F(\sigma_1, J_2)\,[F(I_1, J_2)]^{-1}\right)_b = \delta_{\sigma_1; I_1(b)}$$



$$\left(T_\ell^{\sigma_\ell} P_\ell^{-1}\right)_{a;b} = \left(F(I_{\ell-1}, \sigma_\ell, J_{\ell+1})\,[F(I_\ell, J_{\ell+1})]^{-1}\right)_{a;b} = \delta_{I_{\ell-1}(a), \sigma_\ell; I_\ell(b)}$$

if $I_{\ell-1}(a) \oplus \sigma_\ell \in I_\ell$ — nesting condition



$$\left(T_1^{\sigma_1} P_1^{-1} T_2^{\sigma_2} P_2^{-1} \dots T_\ell^{\sigma_\ell} P_\ell^{-1}\right)_b = \delta_{\sigma_1 \sigma_2 \dots \sigma_\ell; I_\ell(b)}$$

if nesting condition holds for all substrings $\sigma_1 \dots \sigma_\ell$,

Nesting conditions → interpolation property

Haldane Model

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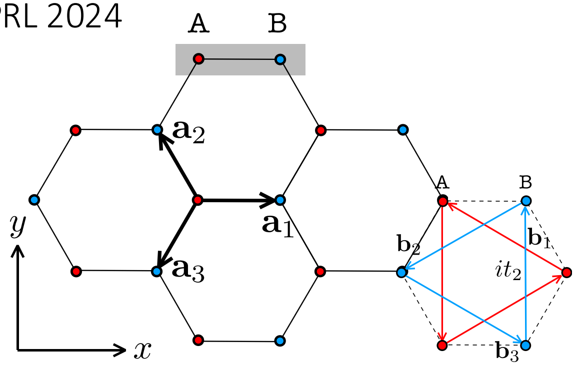
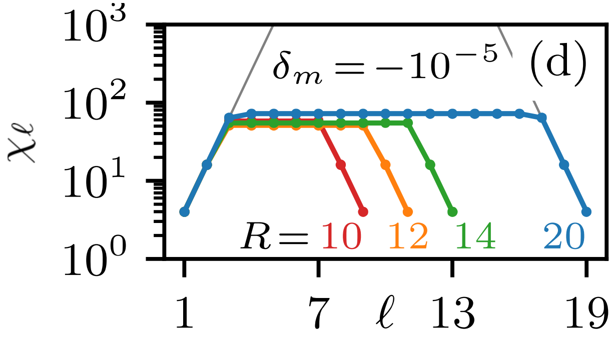
Ritter, Nuñez Fernández, Wallerberger, von Delft, Shinaoka, Waintal, PRL 2024

$$H(\mathbf{k}) = \sum_{i=1}^{3}[\tau^1 \cos(\mathbf{k}\cdot\mathbf{a}_i) + \tau^2 \sin(\mathbf{k}\cdot\mathbf{a}_i)] + \tau^3[m - 2t_2 \sum_{i=1}^{3} \sin(\mathbf{k}\cdot\mathbf{b}_i)]$$

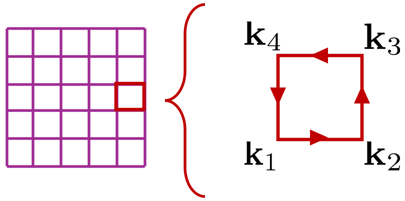
with Pauli matrices $\tau^{1,2,3}$



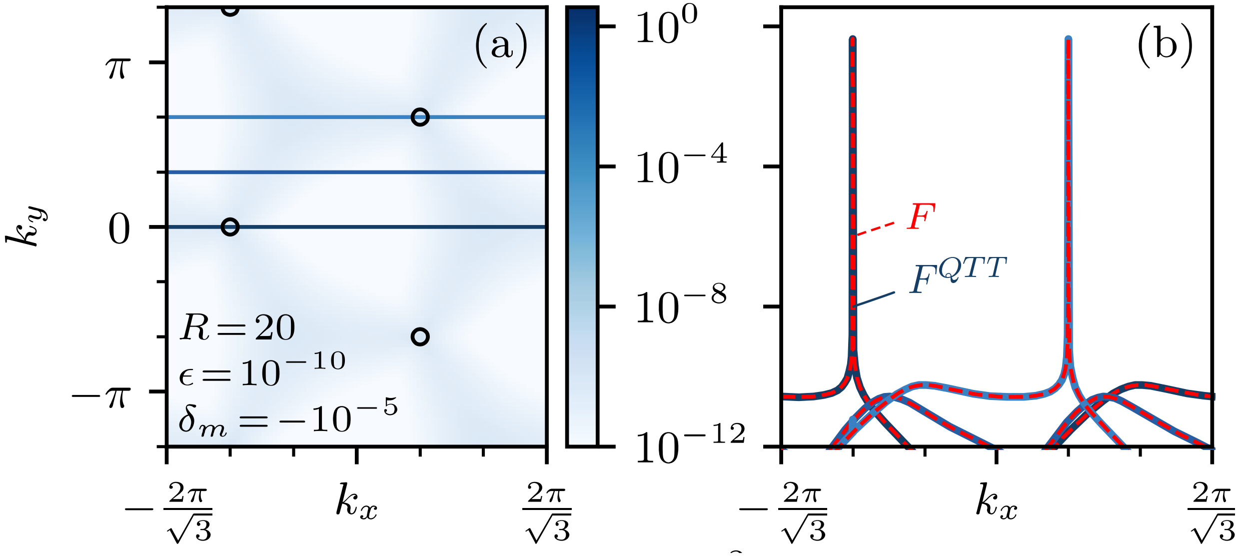


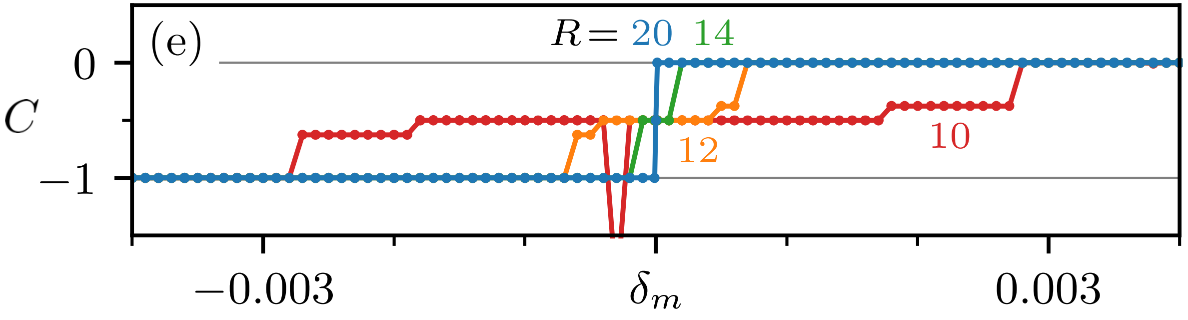
Berry flux

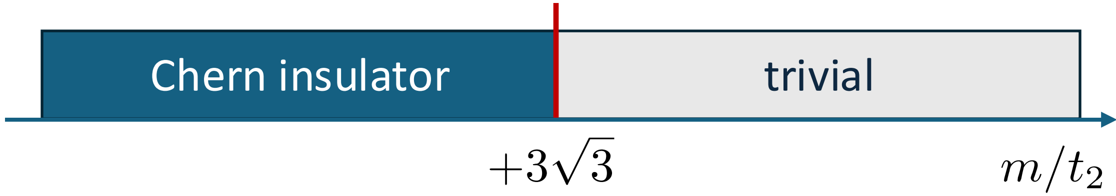
$$F(\mathbf{k}) = -i \arg \prod_{j\in\mathbb{Z}_4} \langle\psi(\mathbf{k}_j)|\psi(\mathbf{k}_{j+1})\rangle$$



$$C \approx \frac{1}{2\pi i}\sum_{\mathbf{k}\in\mathrm{BZ}} F(\mathbf{k})$$







Superfast Fourier transforms

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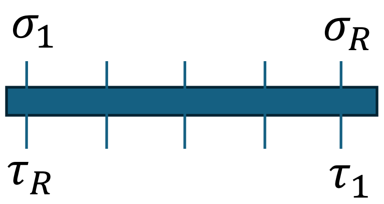
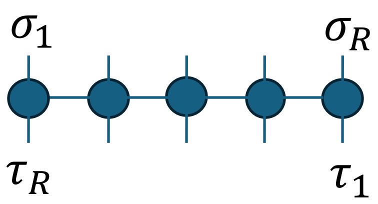
$$F = \exp\left(-\frac{2\pi i}{N}\, k\, x\right) = \exp\left(-i\pi \sum_{\ell,\ell'=1}^{n} 2^{R-\ell-\ell'}\tau_\ell \sigma_{\ell'}\right) =$$



$\approx$



Quantum Fourier Transform Has Small Entanglement

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Direct interpolative construction of the discrete Fourier transform as a matrix product operator

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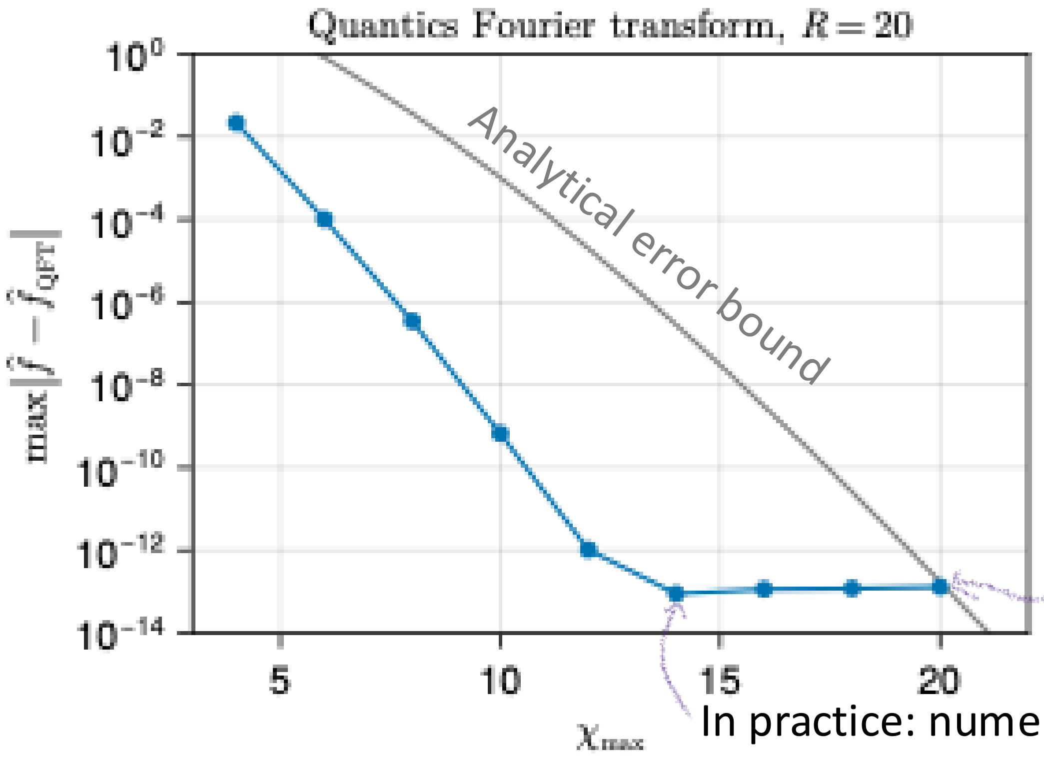
April 5, 2024



Proven: numerical accuracy with $\chi = 20$

In practice: numerical accuracy with $\chi = 14$

Matrix Cross Interpolation

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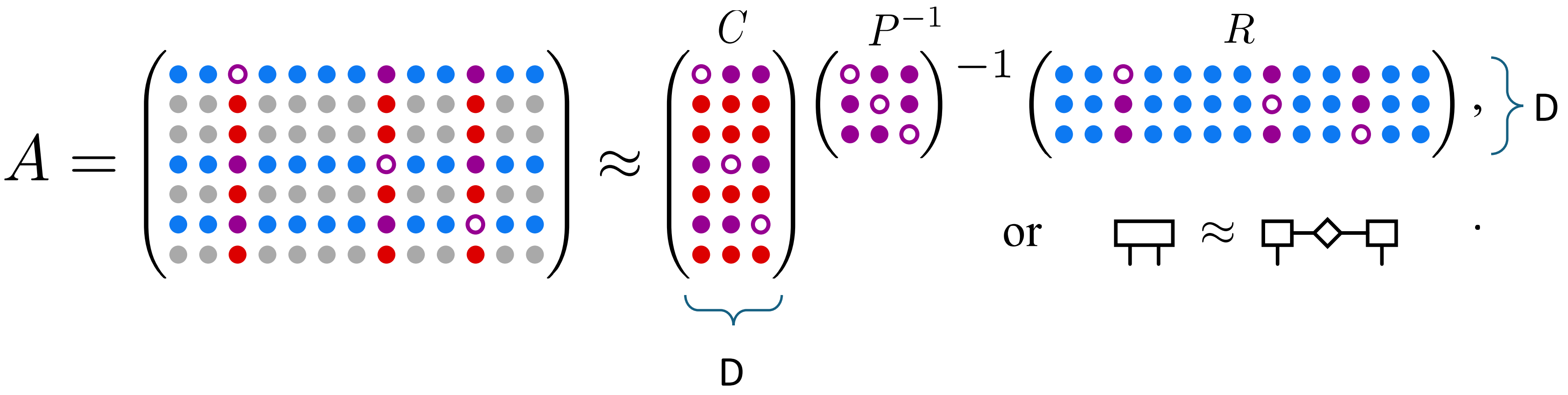
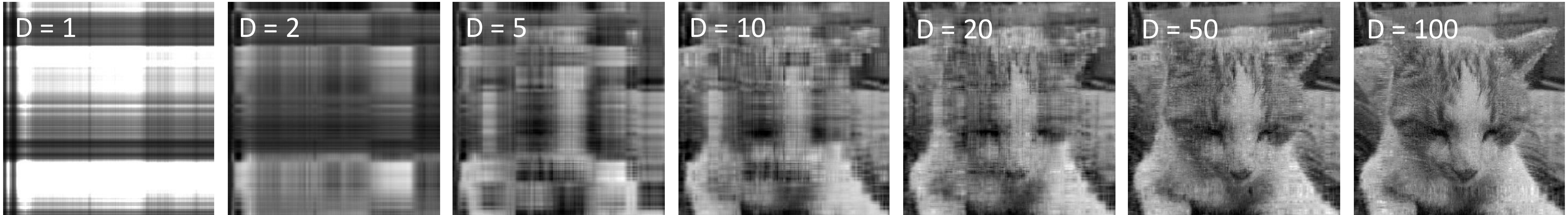
J. Schneider, Journal of Approximation Theory **162**, 1685 (2010).
S. A. Goreinov and E. E. Tyrtyshnikov, Dokl. Math. **83**, 374 (2011).



- Only slightly ($O(D^2)$) less efficient than singular value decomposition.
- Interpolation: Exact representation of pivot rows & columns.
- Example: photo, original size 2048×2048 pixels



Sweeping algorithm

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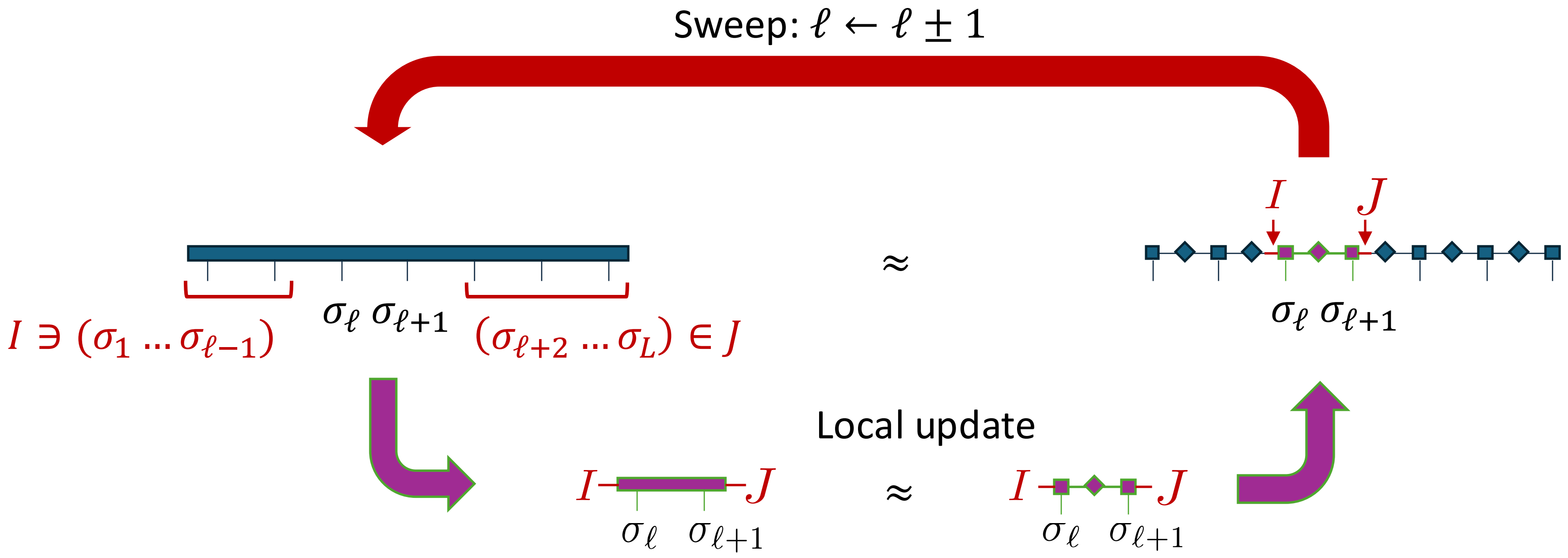
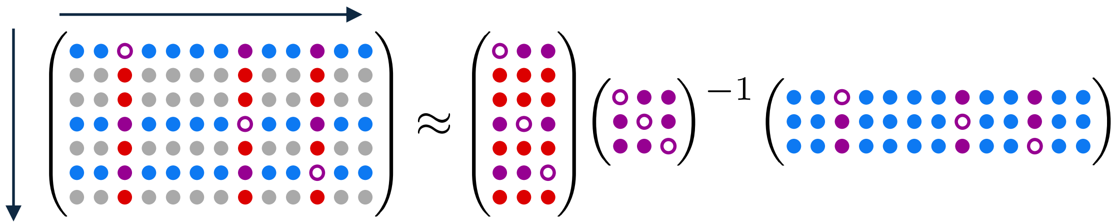
Oseledets, SIAM J Sci Comput 2011; Oseledets, Tyrtyshnikov, Lin Alg Appl 2011;
Nuñez Fernández Jeannin, Dumitrescu, Kloss, Kaye, Parcollet, Waintal, PRX 2022



$$\Pi_{i\sigma_\ell;\sigma_{\ell+1}j} \approx \sum_{ab} T_{i\sigma_\ell;a} P^{-1}_{ab} T_{b;\sigma_{\ell+1}j}$$



Matrix Cross Interpolation

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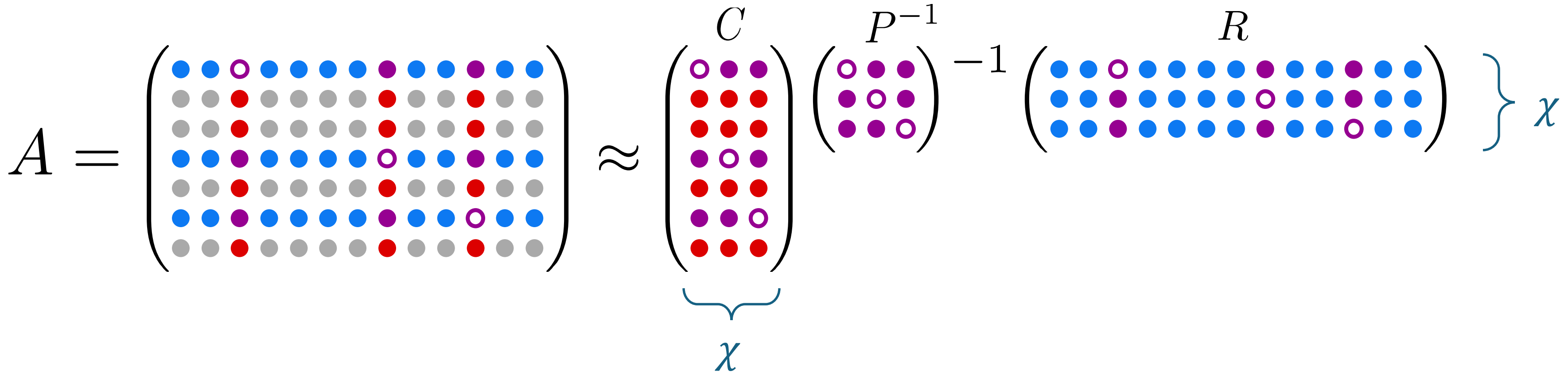
Schneider, J Approx Th 2010; Goreinov & Tyrtyshnikov, Dokl Math 2011



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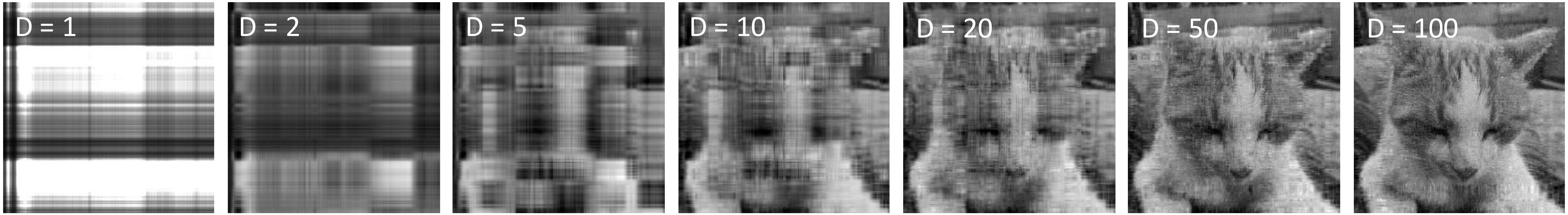

L(D)U factorization

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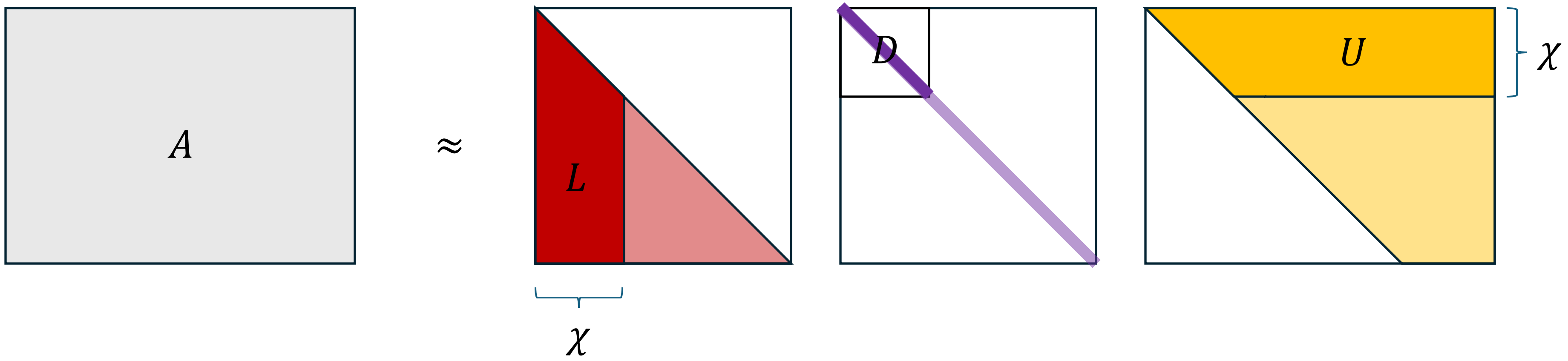
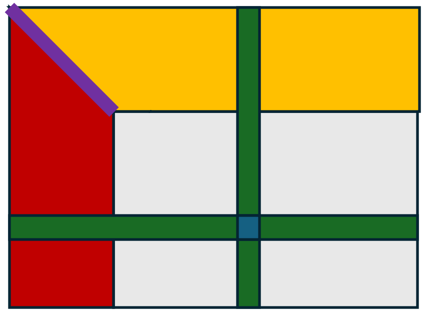
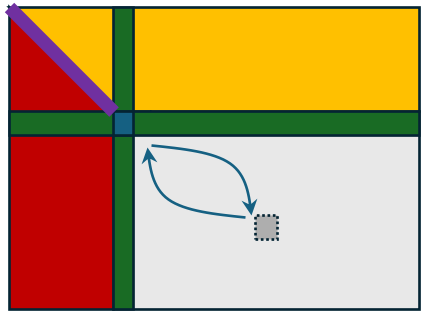


Gaussian elimination

1. Find new pivot



2. Swap rows & columns



3. Elimination

4. Iterate until next pivot < tolerance, or $\chi = \chi_{\max}$

How to find good pivots

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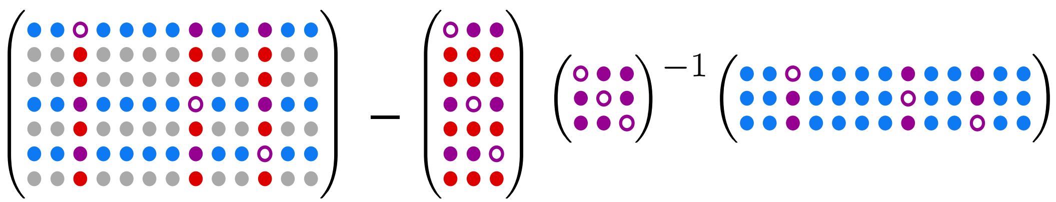
What is a good pivot?

$A - \tilde{A} =$ 

$= A - CP^{-1}R$

$= [A/P]$ (Schur complement)

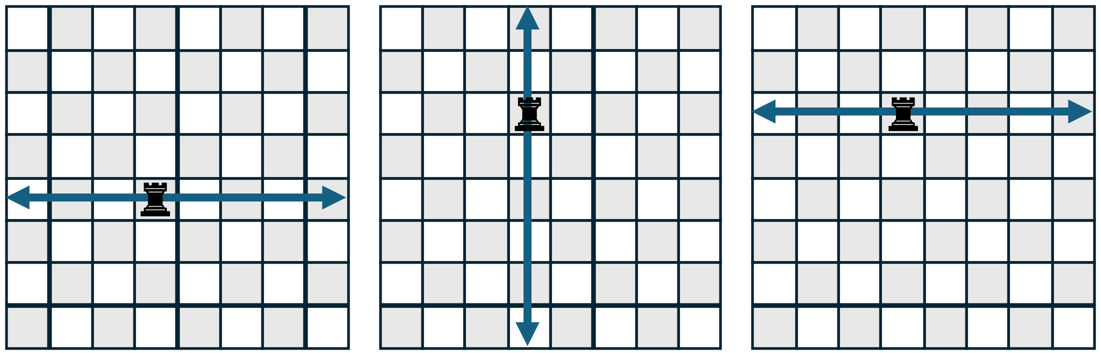
- *Intuitively:*
  Eliminate maximum error $\max(|A - \tilde{A}|)$
- *Maxvol principle:*
  $\tilde{A}$ is quasi-optimal if $|\det P|$ is maximal
  Goreinov and Tyrtyshnikov, Dokl. Math. 2011

Algorithms

Full search $O(mn)$

1. Generate $[A/P]$
2. Look for maximum

Rook search $O(\max(m,n))$



Block rook search $O(\chi \max(m,n)/\chi)$

1. Get column matrix C
2. Update pivots with full search
3. Get row matrix R
4. Update pivots with full search
5. Iterate 1 – 4 to convergence

CI and LU factorization are equivalent

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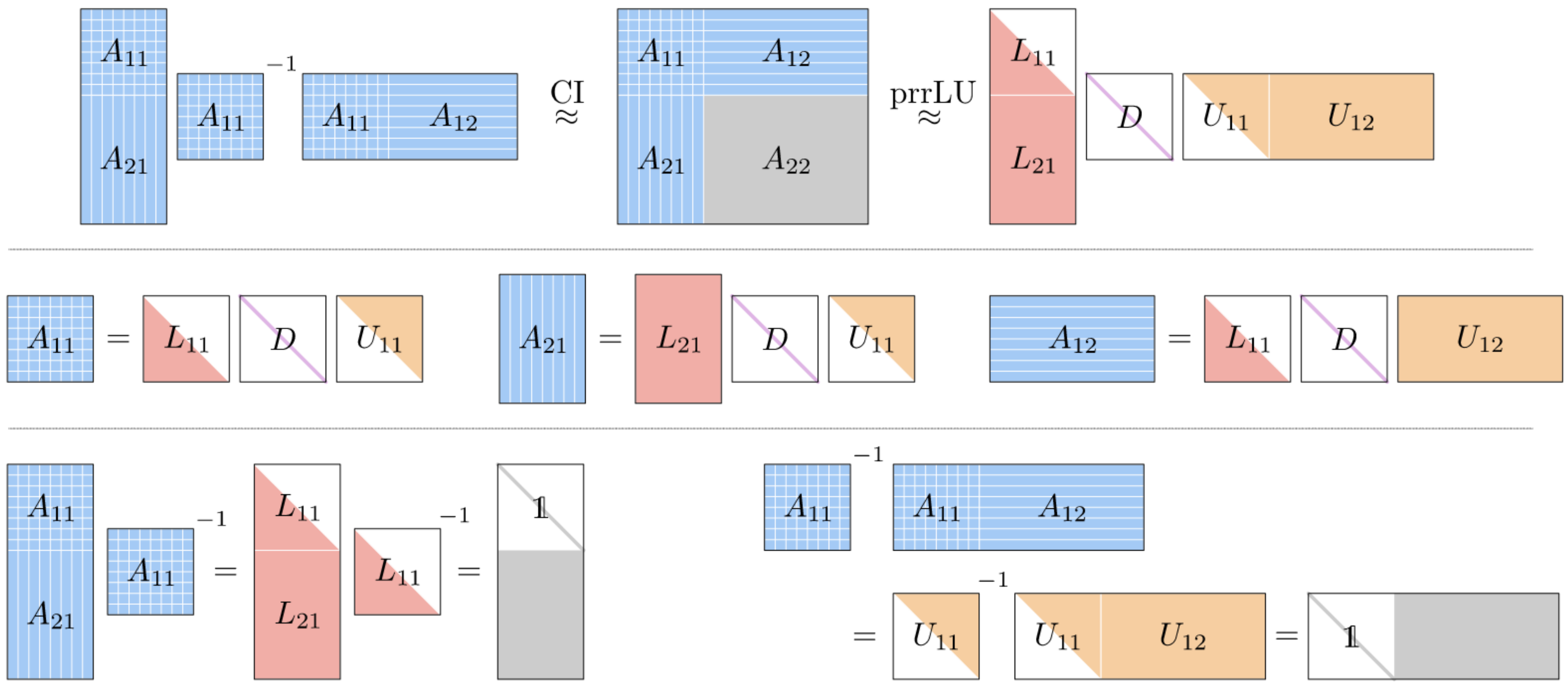

Haldane Model

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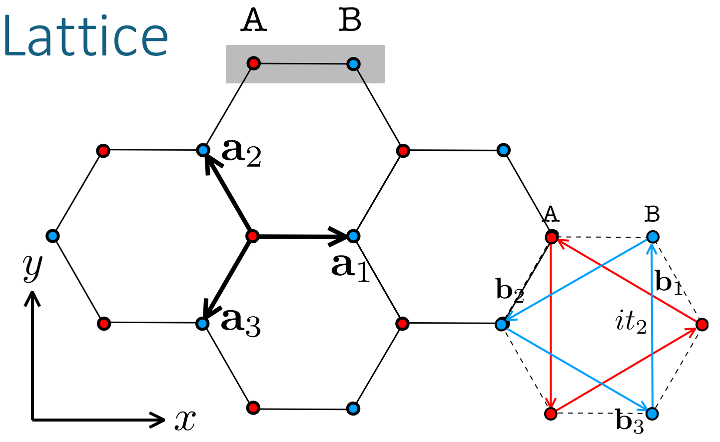
Haldane, PRL 1988

$$H(\mathbf{k}) = \sum_{i=1}^{3}[\tau^1 \cos(\mathbf{k}\cdot\mathbf{a}_i) + \tau^2 \sin(\mathbf{k}\cdot\mathbf{a}_i)] + \tau^3[m - 2t_2 \sum_{i=1}^{3} \sin(\mathbf{k}\cdot\mathbf{b}_i)]$$

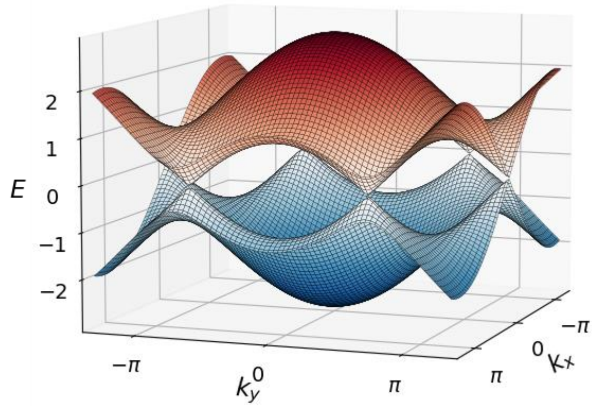
with Pauli matrices $\tau^{1,2,3}$

Lattice



Images: Akhmerov, Sau, van Heck, Ruppert, Skolasiński, Nijholt, Muhammad, and Rosdahl, https://topocondmat.org/w4_haldane/haldane_model.html.

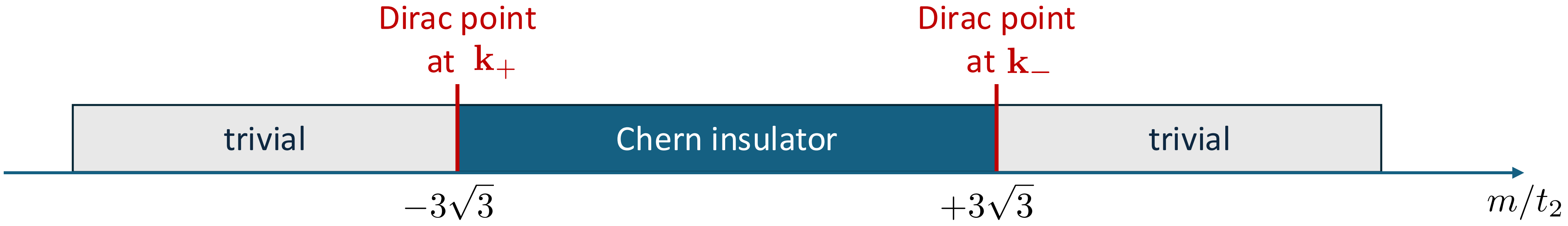
Band structure



At $m = \pm 3\sqrt{3}\cdot t_2$:

- Dirac points at $\mathbf{k}_\mp = \mp \begin{pmatrix} 4\pi/3 \\ 0 \end{pmatrix}$
  (& symmetry-related)
- Topological phase transition between trivial insulator and Chern insulator

Topological phases



Haldane Model

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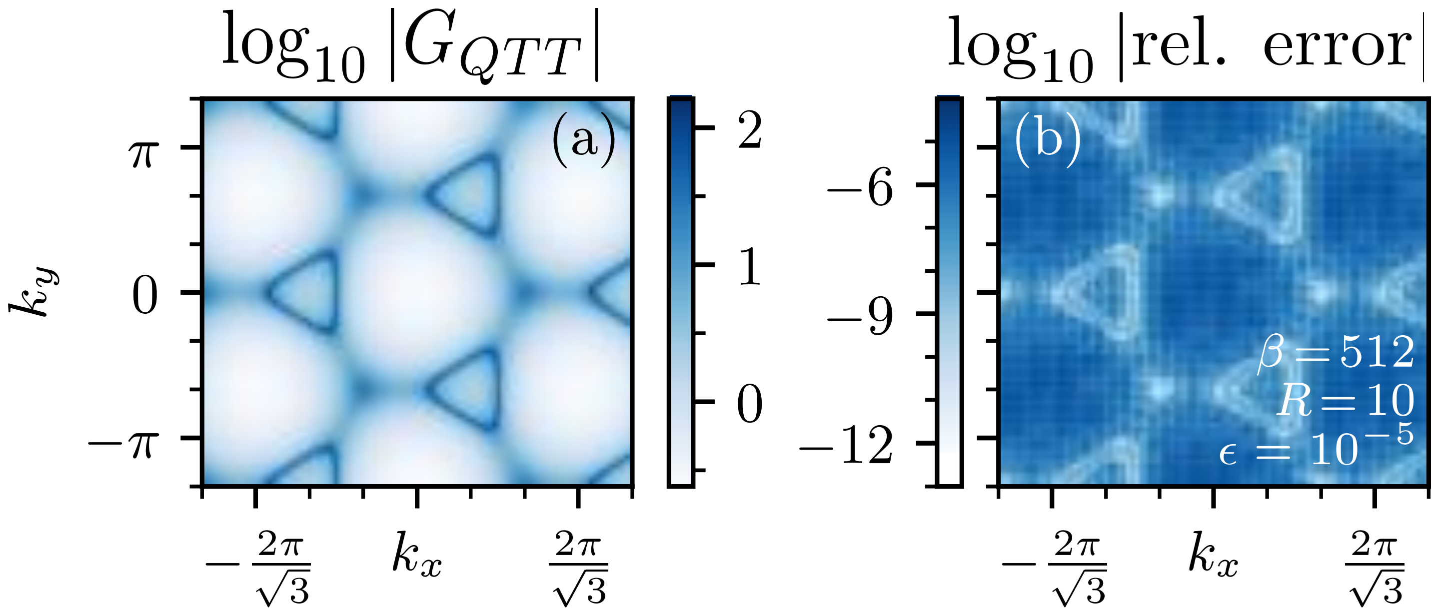
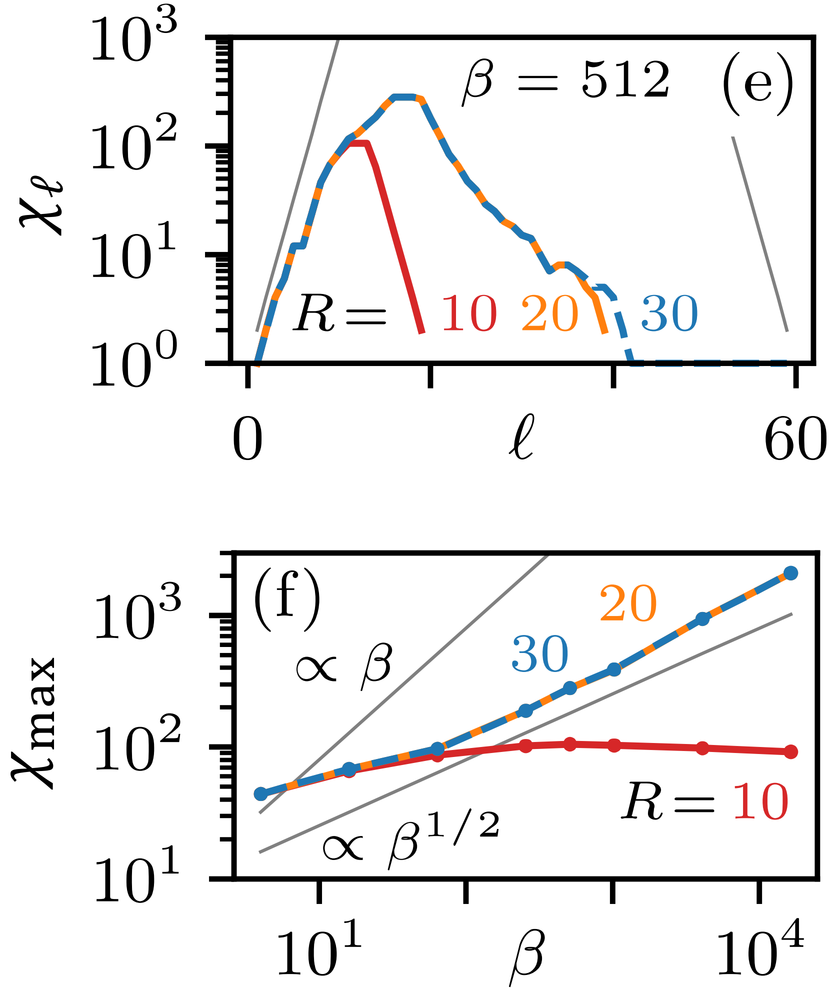
Ritter, Nuñez Fernández, Wallerberger, von Delft, Shinaoka, Waintal: PRL 132 (2024)

Green's function

$$G(\mathbf{k}, i\omega_0) = \mathrm{Tr}\left[\frac{1}{i\omega_0 - H(\mathbf{k}) + \mu}\right]$$

- Structures of width $\omega_0 = \pi/\beta$





Example: 1d oscillating function

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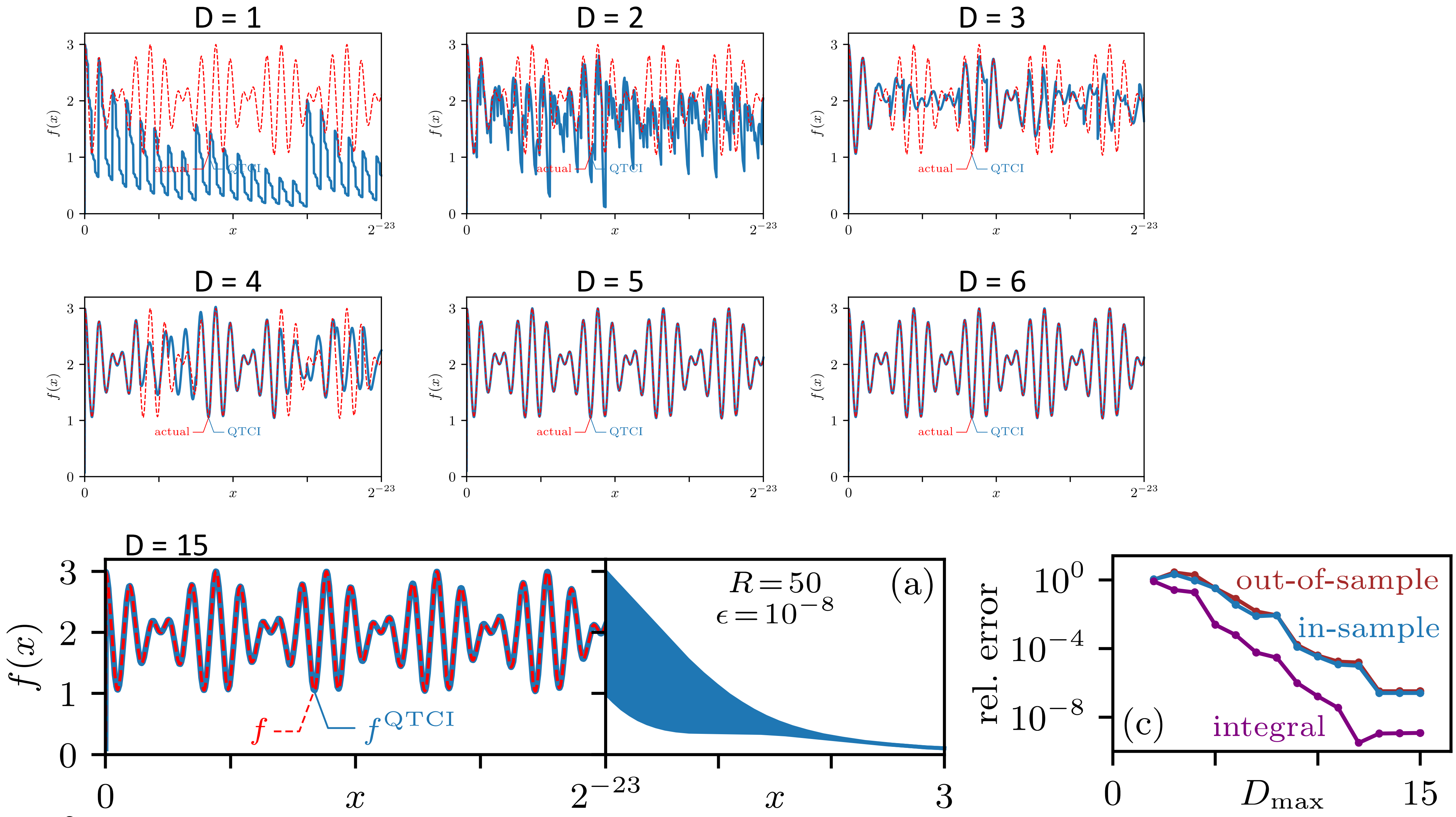

Green’s Function

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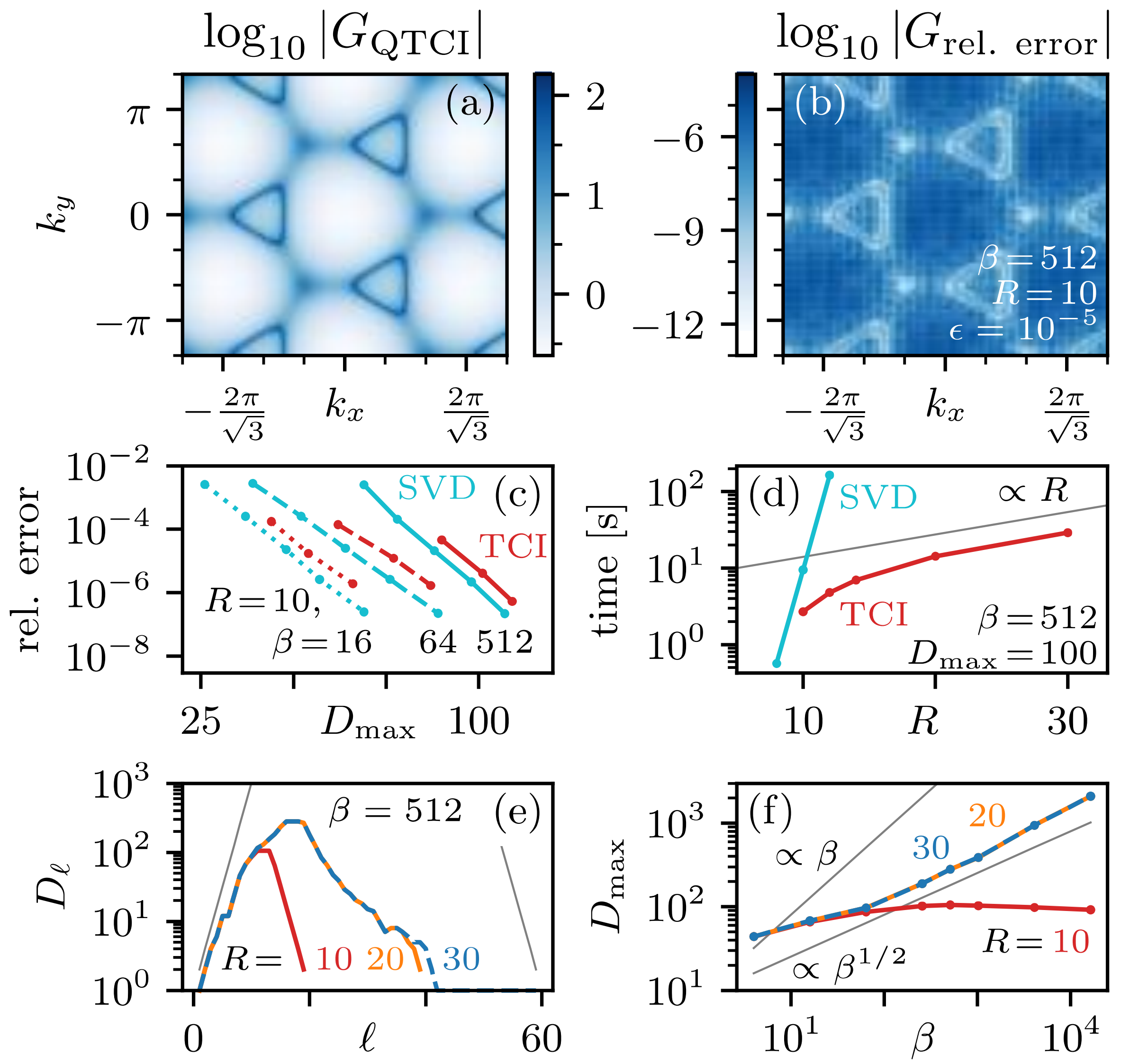
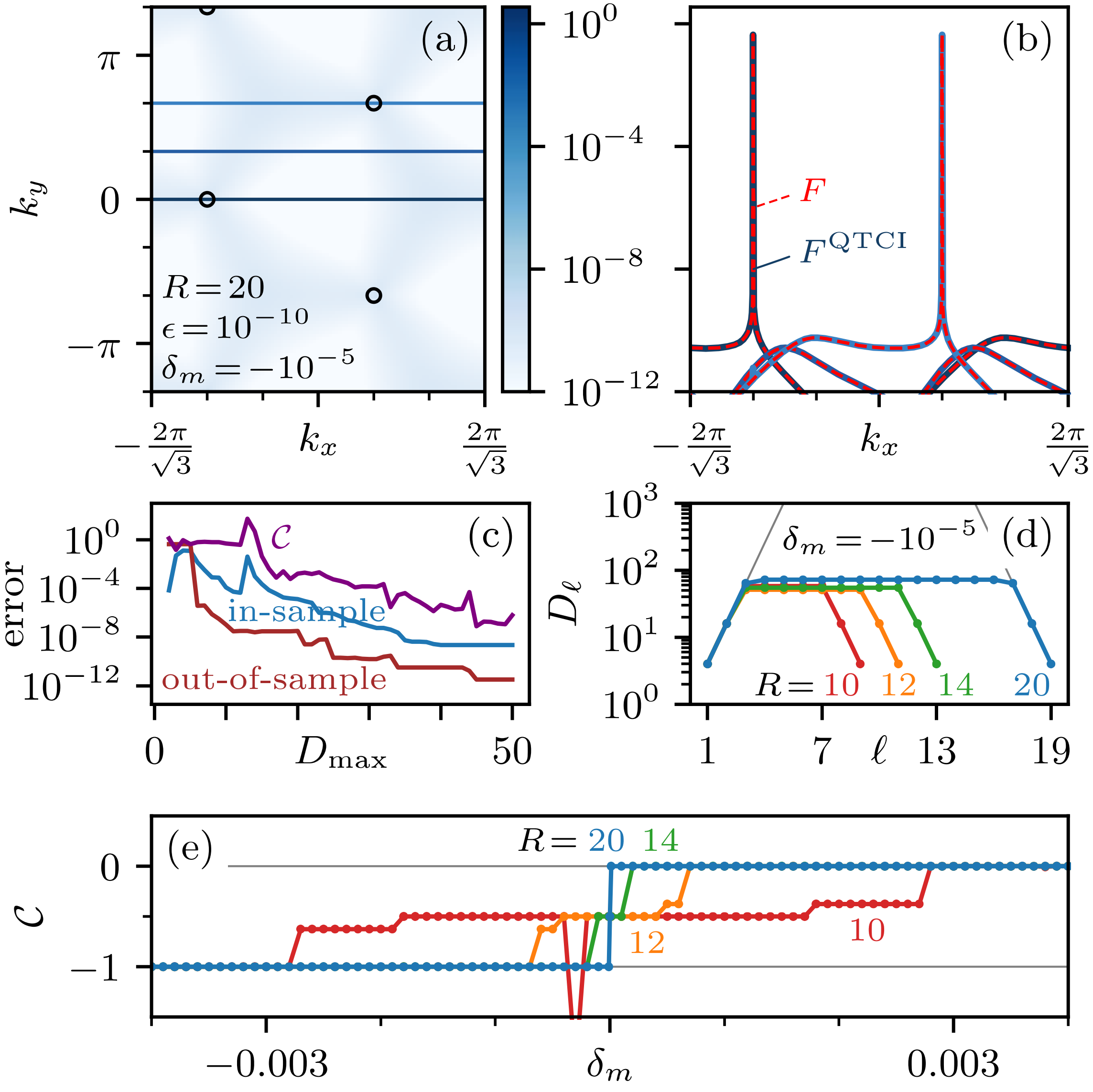
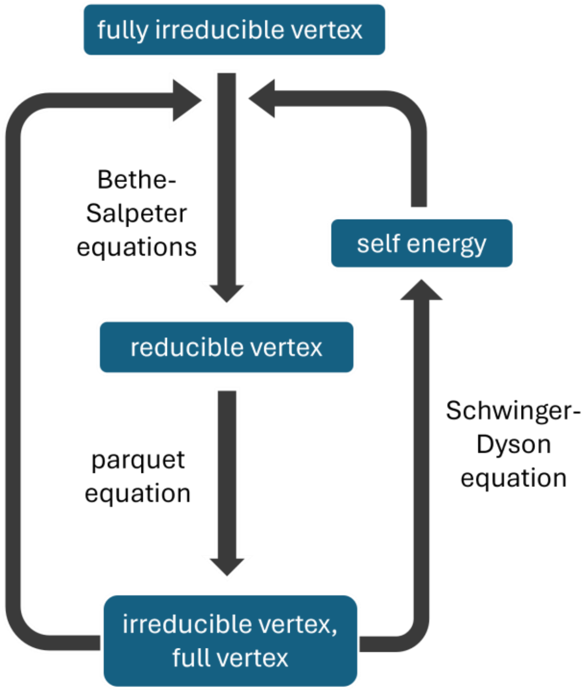
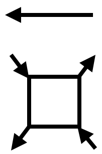


Berry Flux


(a)
k_y
π
0
$-\pi$
$R = 20$
$\epsilon = 10^{-10}$
$\delta_m = -10^{-5}$
$-\frac{2\pi}{\sqrt{3}}$
k_x
$\frac{2\pi}{\sqrt{3}}$
10^{0}
10^{-4}
10^{-8}
10^{-12}
(b)
F
F^{QTCI}
(c)
error
10^{0}
10^{-4}
10^{-8}
10^{-12}
$\mathcal{C}$
in-sample
out-of-sample
0
$D_{\max}$
50
(d)
D_ℓ
10^3
10^2
10^1
10^0
$\delta_m = -10^{-5}$
$R = 10$ 12 14 20
1 7 ℓ 13 19
(e)
$\mathcal{C}$
0
-1
$R = 20$ 14
12
10
-0.003
δ_m
0.003

QTT-Parquet

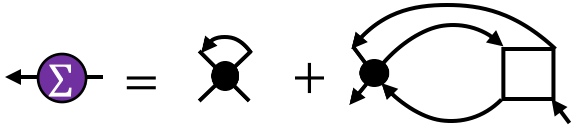




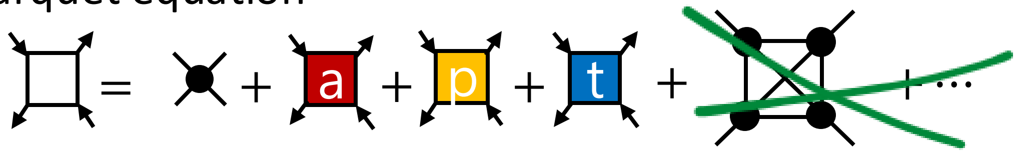
$G(\nu)$

$\Gamma(\nu_1, \nu_2, \nu_3, \nu_4)$

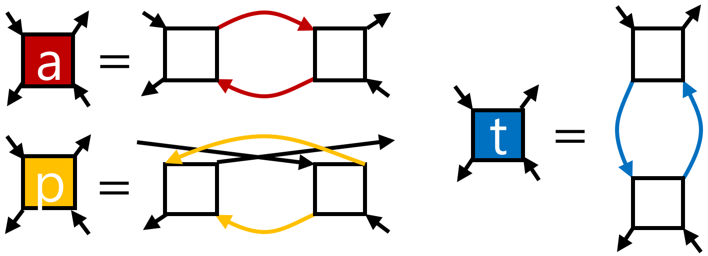
Schwinger-Dyson equation



Parquet equation



Bethe-Salpeter equations



$$F_{d/m}^{\nu\nu'\omega} = \Gamma_{d/m}^{\nu\nu'\omega} - \frac{1}{\beta^2} \sum_{\nu_1\nu_2} \Gamma_{d/m}^{\nu\nu_1\omega} \chi_{0,ph}^{\nu_1\nu_2\omega} F_{d/m}^{\nu_2\nu'\omega}$$